

Geometrical aspects of the Lie Algebra S-expansion Procedure

M. Artebani², R. Caroca³, M. C. Ipinza¹, D. M. Peñafiel¹, P. Salgado¹

¹*Departamento de Física, Universidad de Concepción Casilla 160-C, Concepción, Chile.*

²*Departamento de Matemática, Universidad de Concepción Casilla **160-C**, Concepción, Chile. and*

³*Departamento de Matemática y Física Aplicadas, Universidad Católica de la Santísima Concepción, Alonso de Rivera 285, Concepción, Chile.*

Abstract

In this article it is shown that S-expansion procedure affects the geometry of a Lie group, changing it and leading us to the geometry of another Lie group with higher dimensionality. Is outlined, via an example, a method for determining the semigroup, which would provide a Lie algebra from another. Finally, it is proved that a Lie algebra obtained from another Lie algebra via S-expansion is a non-simple Lie algebra.

I. INTRODUCTION

In Ref. [1] was pointed out that if two physical theories are related by a limiting process, then the associated invariance groups should also be related by some limiting process. This idea was studied in Ref. [2] and introduced the so-called Inönü-Wigner contractions procedure.

Expansions of Lie algebras are a generalization of the contraction method and were introduced some years ago in Refs. [3], [4], [5], [6]. These procedure have been successfully applied in obtaining new Lie algebras and the construction of gravitational theories [8], [9].

The procedure developed in references [3], [4] consists of looking at the algebra \mathcal{G} as described by the Maurer-Cartan forms on the manifold of its associated group G and, after rescaling some of the group parameters by a factor λ , expanding the Maurer-Cartan forms as series in λ . The expansion method, is different from the Inönü-Wigner contraction method albeit, when the algebra dimension does not change in the process, it may lead to a simple Inönü-Wigner or Inönü-Wigner generalized contraction in the sense of Weimar-Woods [12], [13].

On the other hand the method developed in references [5], [6] is a natural outgrowth of the expansion method of Ref. [4]. The procedure is based on combining the structure constant of the algebra with the inner law of a semigroup in order to define the Lie bracket of a new S -expanded algebra. This Abelian Semigroup Expansion method, “S-expansion”, reproduces the results of the Maurer-Cartan forms power series expansion for a particular choice of the semigroup, but is formulated using the Lie algebra generators rather than the associated Maurer-Cartan forms.

These methods appeared to be powerful tools in order to find non-trivial relations between different Lie algebras. The discovery of these relations presents in itself a very interesting problem from both physical and mathematical points of view [7], [8], [9].

In this work it is shown that S-expansion procedure affects the geometry of a Lie group. It is found how changing the magnitude of a vector and the angle between two vectors. Is outlined, via an example, a method for determining the semigroup, which would provide a Lie algebra from another. Finally, it is proved that a Lie algebra obtained from another Lie algebra via S-expansion is a non-simple Lie algebra.

The paper is organized as follows: In Sec. II we review some concepts of the theory

of Lie algebras and the main aspects of the S-expansion procedure. In Sec. *III* we study how the S-expansion procedure affects the geometry of a Lie group. It is found how changing the magnitude of a vector and the angle between two vectors. In section *IV* is outlined, via an example, a method for determining the semigroup, which would provide a Lie algebra from another. In section *V* it is proved that a Lie algebra obtained from another Lie algebra via S-expansion is a non-simple Lie algebra. Here and in the following we have considered finite dimensional Lie algebras and $K = \mathbb{C}$ or $K = \mathbb{R}$ as the fields involved.

II. REVIEW SOME ASPECTS OF LIE ALGEBRAS AND THE S-EXPANSION PROCEDURE

A. Some aspects of Lie Algebras

A Lie algebra is a linear vector space, but because of the group structure on the manifold it inherits a rich algebraic structure. A Lie algebra \mathcal{G} is a vector space over a field K on which a product $[\cdot, \cdot]$, called the Lie bracket, is defined, with the properties

$$\text{If } X, Y \in \mathcal{G}, \text{ then } [X, Y] \in \mathcal{G} \quad (1)$$

$$[\alpha X + \beta Y, Z] = \alpha [X, Z] + \beta [Y, Z] \text{ for } \alpha, \beta \in K \text{ and } X, Y, Z \in \mathcal{G}. \quad (2)$$

$$[X, \alpha Y + \beta Z] = \alpha [X, Y] + \beta [X, Z] \text{ for } \alpha, \beta \in K \text{ and } X, Y, Z \in \mathcal{G}. \quad (3)$$

$$[X, X] = 0 \text{ for all } X \in \mathcal{G}. \quad (4)$$

$$[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0. \quad (5)$$

The property (4) is called skew symmetry and (5) is known as Jacobi identity.

If $\{X_i\}$ is a basis for \mathcal{G} then we have

$$[X_i, X_j] = C_{ij}^k X_k, \quad (6)$$

for some set of constants C_{ij}^k called the structure constants of the algebra. Accordingly, a Lie algebra may be specified by giving a set of constants C_{ij}^k such that

$$C_{ij}^k = -C_{ji}^k \quad (7)$$

$$C_{ij}^m C_{mk}^r + C_{jk}^m C_{mi}^r + C_{ki}^m C_{mj}^r = 0 \quad (8)$$

Definition: A representation of a Lie algebra \mathcal{G} on a vector space V is a mapping ρ from \mathcal{G} to the linear transformation of V such that

$$\rho(\alpha X + \beta Y) = \alpha \rho(X) + \beta \rho(Y) \quad (9)$$

$$\rho([X, Y]) = [\rho(X), \rho(Y)] \quad (10)$$

Transformation of basis: Equations (1) to (5) do not uniquely determine the infinitesimal operators of a given group. We are still free to replace the basis Y_i by another. In fact, under a change of basis transformation

$$X_i = A_i^r Y_r \quad (11)$$

we find that the structure constants change as

$$C_{rs}^{\prime \ t} = (A^{-1})_r^i (A^{-1})_s^j C_{ij}^k A_k^t \quad (12)$$

Let \mathcal{G} be a Lie algebra over the real numbers \mathbb{R} or the complex numbers \mathbb{C} . Consider the linear map adX of \mathcal{G} into itself defined by

$$adX(Y) \equiv [X, Y], \quad X, Y \in \mathcal{G} \quad (13)$$

Using the Jacobi identity (5), we get

$$adX([Y, Z]) = [adX(Y), Z] + [Y, adX(Z)] \quad (14)$$

i.e., the map adX represents a derivation of \mathcal{G} . Furthermore, using (13) and the Jacobi identity we obtain

$$ad[X, Y](Z) = [adX, adY](Z) \quad (15)$$

Hence the set $\mathcal{G}_a = \{adX, X \in \mathcal{G}\}$ is a linear Lie algebra, which is a subalgebra of the Lie algebra \mathcal{G}_A of all derivations and is called the *adjoint algebra*. The map $\psi : X \longrightarrow adX$ is the homomorphism of \mathcal{G} onto \mathcal{G}_a .

It is easily verified that $\psi : X \longrightarrow adX$ is a representation of the Lie algebra \mathcal{G} with \mathcal{G} itself considered as the vector space of the representation. One need only check that $ad[X, Y] = [adX, adY]$, which is a simple consequence of the Jacobi identity.

1. *Adjoint representation*

A better way to look at a change of basis transformation is to determine how the change of basis affects the commutator of an arbitrary element Z in the algebra

$$[Z, X_i] = R(Z)_i^j X_j \quad (16)$$

Under the change of basis (11) we find

$$\begin{aligned} [Z, Y_r] &= S(Z)_r^s Y_s \\ [Z, Y_r] &= \left[Z, (A^{-1})_r^i X_i \right] = (A^{-1})_r^i [Z, X_i] \\ S(Z)_r^s Y_s &= (A^{-1})_r^i R(Z)_i^j X_j = (A^{-1})_r^i R(Z)_i^j A_j^s Y_s \end{aligned}$$

where

$$S(Z)_r^s = (A^{-1})_r^i R(Z)_i^j A_j^s \quad (17)$$

In this manner the effect of a change of basis on the structure constants is reduced to a study of similarity transformations.

The association of a matrix $R(Z)$ with each element of a Lie algebra is called the regular or adjoint representation

$$Z \xrightarrow[\text{representation}]{\text{regular}} R(Z). \quad (18)$$

For example, the representation adX , called the adjoint representation, always provides a matrix representation of the algebra. If $\{X_i\}$ is a basis for \mathcal{G} then

$$adX_i(Y_j) = [X_i, Y_j] = R(X_i)_j^k X_k = C_{ij}^k X_k$$

Therefore the matrix associated with the transformation adX_i is given by

$$R(X_i)_j^k = C_{ij}^k$$

2. *Killing-Cartan inner product*

The Killing-Cartan form of a Lie algebra is a symmetric bilinear form given by

$$\begin{aligned}
(X, X) &= \text{tr} (R(X) R(X)) = \text{tr} (v^i R(X_i) v^j R(X_j)) \\
&= v^i v^j R(X_i)_k^l R(X_j)_l^k = v^i v^j (C_i)_k^l (C_j)_l^k.
\end{aligned} \tag{19}$$

where $X = v^i X_i$. This means that v^i are coordinates in the algebra, which fully define any arbitrary vector.

The inner product of Killing-Cartan provides information about the geometry of the manifold of the group in a neighborhood of identity. The information is obtained in terms of compactness, not compact or nilpotency of the group. This information can be extrapolated to the rest of the manifold using the fact that a Lie group is a "geodesically complete" manifold, i.e., we can reconstruct it completely through the process of exponentiation of algebra.

The vector space of the Lie algebra can be divided into three subspaces under the Cartan-Killing inner product. The inner product is positive-definite, negative-definite, and identically zero. These three subspaces are denoted by:

$$\mathcal{G} = V_- + V_+ + V_0$$

The subspace V_0 is a subalgebra of \mathcal{G} . It is the largest nilpotent invariant subalgebra of \mathcal{G} . Under exponentiation, this subspace maps onto the maximal nilpotent invariant subgroup in the original Lie group.

The subspace V_- is also a subalgebra of \mathcal{G} . It consists of compact (a topological property) operators. That is to say, the exponential of this subspace is a subset of the original Lie group that is parameterized by a compact manifold. It also forms a subalgebra in \mathcal{G} (not invariant).

Finally, the subspace V_+ is not a subalgebra of \mathcal{G} . It consists of noncompact operators. The exponential of this subspace is parameterized by a noncompact submanifold in the original Lie group.

The "division" of algebra in these subspaces implies that in the quadratic form (X, X) of the group, there are summands with different signs representing the spaces denoted as V_- , V_+ and V_0 .

3. Character of an algebra

The character of an algebra, denoted with the symbol χ measures the degree of compactness of the manifold of the a group within a limited range of integer values. The character of an algebra is defined as follows [18]:

$$\chi = \begin{pmatrix} \text{number of} \\ \text{non-compact} \\ \text{generators} \end{pmatrix} - \begin{pmatrix} \text{number of} \\ \text{compact} \\ \text{generators} \end{pmatrix}. \quad (20)$$

which is the trace of the normalized Cartan–Killing form.

Consider a complex and semisimple Lie algebra (although it may contain sets of nilpotent generators) \mathcal{G}_c decomposed as follows

$$X = \sum_{i=1}^l \mathcal{C}^i H_i + \sum_{\alpha \neq 0}^{n-l} c^\alpha E_\alpha \quad (21)$$

where $X \in \mathcal{G}_c$ and \mathcal{C} is a complex coefficient. Here we have a decomposition of $\mathcal{G}_c = \mathbf{H} \oplus \mathbf{E}$ type, where \mathbf{H} is a compact subalgebra \mathcal{G}_c (maximal compact subalgebra). This expression is called "Cartan decomposition" and basically split the algebra in two subspaces, one with a negative definite Killing-Cartan metric and the other positive definite (for semisimple case).

B. The S-expansion procedure

The expansion method proposed in Refs. [3], [4] consists in considering the original algebra as described by its associated Maurer- Cartan forms on the group manifold. Some of the group parameters are rescaled by a factor λ , and the Maurer-Cartan forms are expanded as a power series in λ . This series is finally truncated in a way that assures the closure of the expanded algebra.

Consider now the main aspects of the S -expansion procedure and their properties introduced in Ref. [5]. Let $S = \{\lambda_\alpha\}$ be an abelian, discrete and finite semigroup with 2-selector $K_{\alpha\beta}^\gamma$ defined by

$$K_{\alpha\beta}^\gamma = \begin{cases} 1 & \lambda_\alpha \lambda_\beta = \lambda_\gamma \\ 0 & \text{otherwise,} \end{cases} \quad (22)$$

and \mathfrak{g} a Lie (super)algebra with basis $\{\mathbf{T}_A\}$ and structure constant C_{AB}^C ,

$$[\mathbf{T}_A, \mathbf{T}_B] = C_{AB}^C \mathbf{T}_C. \quad (23)$$

Then it may be shown that the product $\mathfrak{G} = S \times \mathfrak{g}$ is also a Lie (super)algebra with structure constants $C_{(A,\alpha)(B,\beta)}^{(C,\gamma)} = K_{\alpha\beta}^\gamma C_{AB}^C$,

$$[\mathbf{T}_{(A,\alpha)}, \mathbf{T}_{(B,\beta)}] = C_{(A,\alpha)(B,\beta)}^{(C,\gamma)} \mathbf{T}_{(C,\gamma)}. \quad (24)$$

The proof is direct and may be found in Ref. [5].

Definition 1 *Let S be an abelian, discrete and finite semigroup and \mathfrak{g} a Lie algebra. The Lie algebra \mathfrak{G} defined by $\mathfrak{G} = S \times \mathfrak{g}$ is called S -Expanded algebra of \mathfrak{g} .*

When the semigroup has a zero element $0_S \in S$, it plays a somewhat peculiar role in the S -expanded algebra. The above considerations motivate the following definition:

Definition 2 *Let S be an abelian semigroup with a zero element $0_S \in S$, and let $\mathfrak{G} = S \times \mathfrak{g}$ be an S -expanded algebra. The algebra obtained by imposing the condition $0_S \mathbf{T}_A = 0$ on \mathfrak{G} (or a subalgebra of it) is called 0_S -reduced algebra of \mathfrak{G} (or of the subalgebra).*

An S -expanded algebra has a fairly simple structure. Interestingly, there are at least two ways of extracting smaller algebras from $S \times \mathfrak{g}$. The first one gives rise to a *resonant subalgebra*, while the second produces reduced algebras. In particular, a resonant subalgebra can be obtained as follow.

Let $g = \bigoplus_{p \in I} V_p$ be a decomposition of g in subspaces V_p , where I is a set of indices. For each $p, q \in I$ it is always possible to define $i_{(p,q)} \subset I$ such that

$$[V_p, V_q] \subset \bigoplus_{r \in i_{(p,q)}} V_r, \quad (25)$$

Now, let $S = \bigcup_{p \in I} S_p$ be a subset decomposition of the abelian semigroup S such that

$$S_p \cdot S_q \subset \bigcup_{r \in i_{(p,q)}} S_p. \quad (26)$$

When such subset decomposition $S = \bigcup_{p \in I} S_p$ exists, then we say that this decomposition is in resonance with the subspace decomposition of g , $g = \bigoplus_{p \in I} V_p$.

The resonant subset decomposition is crucial in order to systematically extract subalgebras from the S -expanded algebra $G = S \times g$, as is proven in the following

Theorem IV.2 of Ref. [5]: Let $g = \bigoplus_{p \in I} V_p$ be a subspace decomposition of g , with a structure described by eq. (25), and let $S = \bigcup_{p \in I} S_p$ be a resonant subset decomposition of the abelian semigroup S , with the structure given in eq. (26). Define the subspaces of $G = S \times g$,

$$W_p = S_p \times V_p, \quad p \in I. \quad (27)$$

Then,

$$\mathfrak{G}_R = \bigoplus_{p \in I} W_p \quad (28)$$

is a subalgebra of $G = S \times g$.

Proof: the proof may be found in Ref. [5].

Definition 3 *The algebra $G_R = \bigoplus_{p \in I} W_p$ obtained is called a Resonant Subalgebra of the S -expanded algebra $G = S \times g$.*

A useful property of the S -expansion procedure is that it provides us with an invariant tensor for the S -expanded algebra $\mathfrak{G} = S \times \mathfrak{g}$ in terms of an invariant tensor for \mathfrak{g} . As shown in Ref. [7] the theorem VII.2 provide a general expression for an invariant tensor for a 0_S -reduced algebra.

Theorem VII.2 of Ref. [5]: Let S be an abelian semigroup with nonzero elements λ_i , $i = 0, \dots, N$ and $\lambda_{N+1} = 0_S$. Let \mathfrak{g} be a Lie (super)algebra of basis $\{\mathbf{T}_A\}$, and let $\langle \mathbf{T}_{A_1} \cdots \mathbf{T}_{A_n} \rangle$ be an invariant tensor for \mathfrak{g} . The expression

$$\langle \mathbf{T}_{(A_1, i_1)} \cdots \mathbf{T}_{(A_n, i_n)} \rangle = \alpha_j K_{i_a \dots i_n}^j \langle \mathbf{T}_{A_1} \cdots \mathbf{T}_{A_n} \rangle \quad (29)$$

where α_j are arbitrary constants, corresponds to an invariant tensor for the 0_S -reduced algebra obtained from $\mathfrak{G} = S \times \mathfrak{g}$.

Proof: the proof may be found in section VII of Ref. [5].

In summary, in Refs. [5], [6], [7] was proposed a natural outgrowth of power series expansion method, which is based on combining the structure constant of the algebra (\mathcal{G}) with the inner law of a semigroup S in order to define the Lie bracket of a new S -expanded algebra.

Theorem 1 of Ref. [4] shows that, in the more general case, the expanded Lie algebra has the structure constants

$$C_{(A,i)(B,j)}^{(C,k)} = \begin{cases} 0 & \text{when } i + j \neq k \\ C_{AB}^C & \text{when } i + j = k \end{cases}$$

where $i, j, k = 0, \dots, N$ correspond to the order of the expansion, and N is the truncation order. These structure constants can also be obtained within the S -expansion procedure. In order to achieve this, one must consider the 0_S -reduction of an S -expanded algebra where S corresponds to the semigroup. The Maurer-Cartan forms power series expansion of an algebra \mathcal{G} , with truncation order N , coincides with the 0_S -reduction of the $S_E^{(N)}$ -expanded algebra (see Ref. [5]). This is of course no coincidence. The set of powers of the rescaling parameter λ , together with the truncation at order N , satisfy precisely the multiplication law of $S_E^{(N)}$. As a matter of fact, we have $\lambda^\alpha \lambda^\beta = \lambda^{\alpha+\beta}$ and the truncation can be imposed as $\lambda^\alpha = 0$ when $\alpha > N$. It is for this reason that one must demand $0_S T_A = 0$ in order to obtain the Maurer-Cartan expansion as an $S_E^{(N)}$ -expansion: in this case the zero of the semigroup is the zero of the field as well.

The S -expansion procedure is valid no matter what the structure of the original \mathcal{G} Lie algebra is, and in this sense it is very general. However, when something about the structure of \mathcal{G} is known, a lot more can be done. As an example, in the context of Maurer Cartan expansion, the rescaling and truncation can be performed in several ways depending on the structure of \mathcal{G} , leading to several kinds of expanded algebras. Important examples of this are the generalized Inönü–Wigner contraction, or the M algebra as an expansion of $osp(32|1)$ (see Refs. [4, 16]). This is also the case in the context of S -expansions. When some information about the structure of \mathcal{G} is available, it is possible to find subalgebras of $\mathcal{G}^{(E)} = S_E^{(N)} \times \mathcal{G}$ and other kinds of reduced algebras. Among other examples we can find the obtention of General relativity from the Maxwell algebras using the 0_s -reduced-resonant procedure defined above and showed in Refs. [8–11]. In this way, all the algebras obtained by the Maurer Cartan expansion procedure can be reobtained. New kinds of S -expanded algebras can also be obtained by considering semigroups different from $S_E^{(N)}$.

III. THE EXPANSION PROCEDURE AND THE GEOMETRY OF A LIE GROUP

A. Expanding the Killing-Cartan metric

The S -expansion procedure considers the product of an abelian, discrete and finite semi-group $S = \{\lambda_1, \dots, \lambda_P\}$ and a Lie algebra \mathcal{G} , which leads to a new Lie algebra generated by the following $N \cdot P$ generators

$$X_{(\alpha,a)} \equiv X_A = \{X_{(1,1)}, X_{(1,2)}, \dots, X_{(1,N)}, \dots, X_{(P,N)}\} \quad (30)$$

where "P" represents the number of elements of the semigroup and N represents the number of generators of the Lie algebra \mathcal{G} .

Now consider the metric of the new space and its intrinsic geometric properties. If X is a vector of the vector space $S \otimes \mathcal{G}$, then we introduce the inner product as the Killing-Cartan product (for details see Appendix A)

$$(X, X)_{S-\exp} \equiv \text{tr} (R(X) R(X)). \quad (31)$$

Therefore, we have

$$\begin{aligned} (X, X)_{S-\exp} &\equiv \text{tr} (R(X) R(X)) = \text{tr} \left(v^{(\alpha,a)} R(X_{(\alpha,a)}) v^{(\beta,b)} R(X)_{(\beta,b)} \right) \\ &= v^{(\alpha,a)} v^{(\beta,b)} R(X_{(\alpha,a)})_{(\gamma,c)}^{(\delta,d)} R(X_{(\beta,b)})_{(\delta,d)}^{(\gamma,c)} \\ &= v^{(\alpha,a)} v^{(\beta,b)} K_{\alpha\gamma}^{\delta} (C_a)_c^d K_{\beta\delta}^{\gamma} (C_b)_d^c \\ &= v^{(\alpha,a)} v^{(\beta,b)} K_{\alpha\gamma}^{\delta} K_{\beta\delta}^{\gamma} \text{tr} (R(X_a) R(X_b)) \\ &= v^{(\alpha,a)} v^{(\beta,b)} K_{\alpha\gamma}^{\delta} K_{\beta\delta}^{\gamma} (X_a, X_b) \end{aligned} \quad (32)$$

Hence we see that the product of Killing-Cartan, undergoes a change due to the presence of the K -selectors.

Because the Killing-Cartan product, of the original algebra appears immersed in the Killing-Cartan product of the expanded algebra, the calculations are simplified. Since a metric is a symmetric bilinear form, we can use the spectral theorem, to obtain the corresponding diagonal metric. (This theorem states that every real symmetric matrix is diagonalizable in \mathbb{R}).

This means that a transformation $X \longrightarrow \tilde{X}$ allows to write $(\tilde{X}, \tilde{X}) = 0, \forall a \neq b$. So that

$$(\tilde{X}, \tilde{X}) = \tilde{v}^a \tilde{v}^a (\tilde{X}_a, \tilde{X}_a),$$

therefore the Killing-Cartan product of the expanded algebra, takes the form

$$(\tilde{X}, \tilde{X})_{S-\exp} = \tilde{v}^{(\alpha,a)} \tilde{v}^{(\beta,a)} K_{\alpha\gamma}^{\delta} K_{\beta\delta}^{\gamma} (\tilde{X}_a, \tilde{X}_a). \quad (33)$$

This means

$$\left(\tilde{X}, \tilde{X}\right)_{S-\text{exp}} = \tilde{v}^{(\alpha,a)} \tilde{v}^{(\beta,b)} K_{\alpha\gamma}^{\delta} K_{\beta\delta}^{\gamma} \left(\tilde{X}_a, \tilde{X}_b\right) \neq 0, \quad \text{when } a = b. \quad (34)$$

This new inner product is invariant under the action of the S-expanded transformations (S-expanded generators) that constituting the new Lie algebra $S \otimes \mathcal{G}$ (A proof of invariance is given in Appendix B). This is because the original inner product (X_a, X_b) is invariant under the \mathcal{G} transformations and the part that involve the K selectors doesn't affect the inner product $v^{(\alpha,a)} v^{(\beta,b)} K_{\alpha\gamma}^{\delta} K_{\beta\delta}^{\gamma} (X_a, X_b)$ because of the abelian and the associativity properties of the semigroup S .

However not all matrices are diagonalizable. The decomposition of a matrix in Jordan canonical form is a decomposition that generalizes the notion of diagonalization. The interesting thing about this decomposition is that every matrix can be taken to its canonical form, i.e., any matrix A can be written in the form

$$A = SJS^{-1},$$

where J , is known as Jordan matrix.

This means that for each value of the index "a", the values that can take the pair of indices $\{\alpha, \beta\}$ will play the role of labeling new coordinates unlike the index set $\{\gamma, \delta\}$ whose role is of geometrical nature. For this reason we use the indices $\{i, j\}$ to denote $\{\alpha, \beta\}$. Thus, we have

$$\begin{aligned} \left(\tilde{X}, \tilde{X}\right)_{S-\text{exp}} &= \sum_a^{\dim(\mathcal{G})} \left[\sum_{i,j,\gamma,\delta}^p \tilde{v}^{(i,a)} \tilde{v}^{(j,a)} K_{i\gamma}^{\delta} K_{j\delta}^{\gamma} \right] \left(\tilde{X}_a, \tilde{X}_a\right) \\ \left(\tilde{X}, \tilde{X}\right)_{S-\text{exp}} &= \sum_a^{\dim(\mathcal{G})} \left[\sum_{i,j}^p \tilde{v}^{(i,a)} \tilde{v}^{(j,a)} \left[\sum_{\gamma,\delta}^p K_{i\gamma}^{\delta} K_{j\delta}^{\gamma} \right] \right] \left(\tilde{X}_a, \tilde{X}_a\right). \end{aligned} \quad (35)$$

If of all the bases that diagonalize the original metric, we choose the basis of eigenvectors, we have

$$\left(\tilde{X}, \tilde{X}\right)_{S-\text{exp}} = \sum_a^{\dim(\mathcal{G})} \left[\sum_{i,j}^P \tilde{v}^{(i,a)} \tilde{v}^{(j,a)} \left[\sum_{\gamma,\delta}^P K_{i\gamma}^\delta K_{j\delta}^\gamma \right] \right] \left(\tilde{X}_a, \tilde{X}_a\right) \quad (36)$$

$$\longrightarrow \sum_a^{\dim(\mathcal{G})} \left[\sum_{i,j}^P \hat{v}^{(i,a)} \hat{v}^{(j,a)} \left[\sum_{\gamma,\delta}^P K_{i\gamma}^\delta K_{j\delta}^\gamma \right] \right] \lambda_a \quad (37)$$

where the change $\tilde{v} \longrightarrow \hat{v}$ means the rotation of the base under $O(n)$ and λ_a corresponds to the eigenvalue associated to the eigenvector X_{λ_a} . Note that although the original metric is diagonal, the expanded metric need not be diagonal. This is because the coordinates of the new $S \otimes \mathcal{G}$ vector space, denoted as $\tilde{v}^{(\alpha,a)}$, has indices of the semigroup S . Thus we have

$$\begin{aligned} (X, X)_{S-\text{exp}} &= \left(\tilde{v}^{(\alpha,a)}\right)_{\times(P \cdot N)} \begin{pmatrix} \lambda_1 \left(\sum_{\gamma,\delta}^p K_{\alpha\gamma}^\delta K_{\beta\delta}^\gamma\right) & & O \\ & \ddots & \\ O & & \lambda_N \left(\sum_{\gamma,\delta}^p K_{\alpha\gamma}^\delta K_{\beta\delta}^\gamma\right) \end{pmatrix} \begin{pmatrix} \tilde{v}^{(\beta,b)} \end{pmatrix}_{\substack{(P \cdot N) \\ \times 1}} \\ &\equiv \left(\tilde{v}^{(\alpha,a)}\right)_{\times(P \cdot N)} \begin{pmatrix} \lambda_1 (M_K)_{p \times p} & & O \\ & \ddots & \\ O & & \lambda_N (M_K)_{p \times p} \end{pmatrix} \begin{pmatrix} \tilde{v}^{(\beta,b)} \end{pmatrix}_{\substack{(P \cdot N) \\ \times 1}} \\ &\equiv \left(\tilde{v}^{(\alpha,a)}\right)_{1 \times (P \cdot N)} (g)_{(\alpha,a)(\beta,b)} \left(\tilde{v}^{(\beta,b)}\right)_{(P \cdot N) \times 1} \end{aligned} \quad (38)$$

From here, we can see that the intrinsic local geometry of the variety of the expanded group depends strongly on the characteristics of the semigroup.

The matrix $(M_K)_{p \times p}$ defined as $(M_K)_{p \times p} = K_{\alpha\gamma}^\delta K_{\beta\delta}^\gamma$, is symmetric because the semigroup is abelian. This allows writing

$$(X, X)_{S-\text{exp}} = \left(\tilde{v}^{(\alpha,a)}\right)_{1 \times (P \cdot N)} \begin{pmatrix} \lambda_1 (M_K)_{p \times p} & & O \\ & \ddots & \\ O & & \lambda_N (M_K)_{p \times p} \end{pmatrix} \begin{pmatrix} \tilde{v}^{(\beta,a)} \end{pmatrix}_{(P \cdot N) \times 1}$$

or,

$$(X, X)_{S-\text{exp}} =$$

$$\begin{pmatrix} \tilde{v}^{(\alpha,a)} \end{pmatrix}_{\times(P \cdot N)}^1 \begin{pmatrix} V.P. \end{pmatrix}_{\times(P \cdot N)}^{(P \cdot N)} \begin{pmatrix} \lambda_1 \bar{\lambda}_1 & & & & \\ & \ddots & & & \\ & & \lambda_1 \bar{\lambda}_p & & \\ & & & \ddots & \\ & & & & \lambda_N \bar{\lambda}_1 \\ & O & & & \ddots \\ & & & & & \lambda_N \bar{\lambda}_P \end{pmatrix}_{\times(P \cdot N)}^{(P \cdot N)} \begin{pmatrix} V.P. \end{pmatrix}_{\times(P \cdot N)}^{-1} \begin{pmatrix} \tilde{v}^{(\beta,a)} \end{pmatrix}_{\times 1}^{(P \cdot N)}$$

where

$$\begin{pmatrix} V.P. \end{pmatrix}_{(p \cdot N) \times (p \cdot N)} = \begin{pmatrix} V.P.(\lambda_1) & \cdots & V.P.(\lambda_{N \cdot p}) \end{pmatrix} \quad (39)$$

is the matrix of eigenvectors associated with the eigenvalues $\bar{\lambda}$ of a symmetric matrix, denoted as $(g)_{AB}$.

This result is very important in, for example, the study of change in the signature of the metric with respect to the signature of the original metric. Since the entries of these submatrices are formed by elements of the form $K_{\alpha\gamma}^\delta K_{\beta\delta}^\gamma$ will always be possible to modify them using the laws of the internal composition of the semigroup S .

To see how it affected the metric of signature $(+1, +2, \dots, +l, \dots, -1, -2, \dots, -m)$ with $l + m = N$, rewrite (g_{AB}) as

$${}_d(g_{AB}) =$$

$$\begin{pmatrix}
+\lambda_1 (\pm \bar{\lambda}_1) & & & & & & & & \\
& \ddots & & & & & & & \\
& & +\lambda_1 (\pm \bar{\lambda}_p) & & & & & & \\
& & & \ddots & & & & & \\
& & & & +\lambda_l (\pm \bar{\lambda}_p) & & & & \\
& & & & & -\lambda_{l+1} (\pm \bar{\lambda}_1) & & & \\
& & & & & & \ddots & & \\
& & O & & & & & -\lambda_{l+1} (\pm \bar{\lambda}_p) & \\
& & & & & & & & \ddots \\
& & & & & & & & & -\lambda_{l+m} (\pm \bar{\lambda}_p)
\end{pmatrix}_{\substack{N \cdot p \times \\ N \cdot p}} \quad (40)$$

so if the matrix $(M_K)_{p \times p}$ has for example a negative eigenvalue $\pm \bar{\lambda}_p \rightarrow -\bar{\lambda}_p$, it is found

$${}_d(g_{AB}) = \begin{pmatrix}
+\lambda_1 (\pm \bar{\lambda}_1) & & & & & & & & \\
& \ddots & & & & & & & \\
& & -\lambda_1 \bar{\lambda}_p & & & & & & \\
& & & \ddots & & & & & \\
& & & & -\lambda_l \bar{\lambda}_p & & & & \\
& & & & & -\lambda_{l+1} (\pm \bar{\lambda}_1) & & & \\
& & & & & & \ddots & & \\
& & O & & & & & +\lambda_{l+1} \bar{\lambda}_p & \\
& & & & & & & & \ddots \\
& & & & & & & & & +\lambda_{l+m} \bar{\lambda}_p
\end{pmatrix}_{\substack{N \cdot p \\ \times N \cdot p}} \quad (41)$$

So there will be a change of sign for each of the N matrices $(M_K)_{p \times p}$. This means that the dimension of the original algebra, as well as the dimension of its subspaces plays a crucial role in the study of the final signature of the metric. If the dimension of the metric is $N = 2n$ for some $n \in \mathbb{N}$ and $p \in \{\mathbb{N}\}$, then necessarily there will be an even number of sign changes in the diagonal: Even numbers form a semigroup under multiplication and similarly for $N = 2n + 1$.

A "change of sign" means that the signs have changed due to the presence of negative eigenvalues in M_K matrix. That is, are changes with respect to the case where the signature of the diagonal depends only on the eigenvalues of the original metric. Denote the number of sign changes in the metric S-expanded with the symbol $\#_-$. We will use the following notation for algebra \mathcal{G} of $\dim(\mathcal{G}) = N$

$$\begin{aligned}
& \text{number of matrices } (M_K)_{p \times p} = N \\
& \text{number of eigenvalues of } g_{ab} = N \\
& \text{number of elements of the semigroup} = P \\
& \text{number of diagonal elements of } {}_d(X, X)_{S-\text{exp}} = N \cdot P \\
& \text{number of negative eigenvalues of } (M_K)_{p \times p} = Q \leq P \\
& \text{Total number of changes of sign} = N \cdot Q.
\end{aligned} \tag{42}$$

In the number of diagonal elements of ${}_d(X, X)_{S-\text{exp}} = N \cdot P$ should consider the zeros that come from the original algebra, which may have nilpotent subalgebras or subspaces.

Consider now the distribution of number $\#_-$ along the diagonal ${}_d(X, X)_{S-\text{exp}}$. To see this it is necessary first to clarify the increase in the dimensionality of each subspace of the vector space of the original \mathcal{G} algebra, which will be differentiated by the corresponding diagonal signature. They will be denoted by V_+ subspaces whose diagonal is totally positive and V_- the subspaces whose diagonal is totally negative. There are also spaces (nilpotent) V_0 whose metric has only zeros.

So we have to add to the above table the quantities

$$\begin{aligned}
& \text{ran}(V_+) = l \\
& \text{ran}(V_-) = m \\
& \text{ran}(V_+)_{S-\text{exp}} = l \cdot P \\
& \text{ran}(V_-)_{S-\text{exp}} = m \cdot P
\end{aligned} \tag{43}$$

This allows us to make a detailed analysis of each of the subspaces of \mathcal{G} . Indeed, it is straightforward to see that

$$\begin{aligned}
\text{change of sign in } (V_+)_{s-\text{exp}} &= l \cdot Q \\
\text{change of sign in } (V_-)_{s-\text{exp}} &= m \cdot Q \\
\text{change of sign in } (\mathcal{G})_{s-\text{exp}} &= (l + m) \cdot Q \\
&= N \cdot Q
\end{aligned}$$

Note that, if $\text{ran}(V_+) = \text{ran}(V_-)$, then

$$\#_-(V_+)_{s-\text{exp}} = l \cdot Q = m \cdot Q = \#_-(V_-)_{s-\text{exp}}$$

and that if $\text{ran}(V_+) \neq \text{ran}(V_-)$, then

$$\#_-(V_+)_{s-\text{exp}} = l \cdot Q \neq m \cdot Q = \#_-(V_-)_{s-\text{exp}}$$

as well as if $\text{ran}(V_+) > \text{ran}(V_-)$ or $\text{ran}(V_+) < \text{ran}(V_-)$, then

$$\begin{aligned}
&\#_-(V_+)_{s-\text{exp}} = l \cdot Q > m \cdot Q = \#_-(V_-)_{s-\text{exp}} \\
\text{or } &\#_-(V_+)_{s-\text{exp}} = l \cdot Q < m \cdot Q = \#_-(V_-)_{s-\text{exp}}
\end{aligned}$$

Now consider a classification of algebras, based on the above results

$$\begin{aligned}
I &= \{\text{ran}(\mathcal{G}) = 2n \text{ con } n \in \mathbb{N} : \text{ran}(V_+) = \text{ran}(V_-)\} \\
II &= \{\text{ran}(\mathcal{G}) = 2n \text{ con } n \in \mathbb{N} : \text{ran}(V_+) \neq \text{ran}(V_-)\} \\
III &= \{\text{ran}(\mathcal{G}) = 2n + 1 \text{ con } n \in \mathbb{N}\} .
\end{aligned} \tag{44}$$

In the study of changes of sign in the expanded metrics, $\text{ran}(\mathcal{G})$ plays an interesting role. In fact, consider the analysis of each of these three sets when the M_K matrix has Q negative eigenvalues:

a. Case I with $\dim(V_0) = 0$: In this case,

$$\begin{aligned}
&\text{ran}(V_+) = l = \text{ran}(V_-) = m, \quad \text{where } l \text{ and } m \text{ are even} \\
&\text{ran}(V_+)_{s-\text{exp}} = l \cdot P = \text{ran}(V_-)_{s-\text{exp}} = m \cdot P \text{ where } l \cdot P \text{ and } m \cdot P \text{ are even}
\end{aligned} \tag{45}$$

$$\begin{aligned}
\text{ran}(\mathcal{G}) &= l + m : \text{even} \\
\text{ran}(\mathcal{G})_{S\text{-exp}} &= (l + m) \cdot P = \text{ran}(\mathcal{G}) \cdot P : \text{even}, \forall P \in \mathbb{N}
\end{aligned} \tag{46}$$

This means that these algebras conserve parity under S -expansion because even numbers have the property of closure under addition and multiplication. In presence of negative eigenvalues producing sign changes it is found

$${}_d(g_{AB})_I = \begin{pmatrix} \Gamma & O \\ O & \Lambda \end{pmatrix}$$

where

$$\begin{aligned}
{}_d(g_{AB})_\Gamma &= \begin{pmatrix} \lambda_{1d}(M_K) & & & & \\ & \ddots & & & \\ & & \ddots & & \\ & & & \lambda_{i\ d}(M_K) & \\ & O & & & \ddots \\ & & & & \lambda_{l+m\ d}(M_K) \end{pmatrix}_{\substack{(l \cdot P) \times (l \cdot P) \\ (m \cdot P) \times (m \cdot P)}} \\
{}_d(g_{AB})_\Lambda &= \begin{pmatrix} -\lambda_{l+1\ d}(M_K) & & & & \\ & \ddots & & & \\ & & -\lambda_{l+i\ d}(M_K) & & \\ & & & \ddots & \\ & O & & & -\lambda_{l+m\ d}(M_K) \end{pmatrix}_{\substack{(m \cdot P) \times (m \cdot P) \\ (l \cdot P) \times (l \cdot P)}}
\end{aligned}$$

where

$${}_d(M_K) = \begin{pmatrix} -\bar{\lambda}_1 & & & & \\ & \ddots & & & \\ & & -\bar{\lambda}_Q & & \\ & & & \bar{\lambda}_{Q+1} & \\ O & & & & \ddots \\ & & & & & \bar{\lambda}_P \end{pmatrix} \tag{47}$$

In this case it is found that for each of the l positive eigenvalues of ${}_d(X, X)$, are produced Q changes of sign of the form $+$ \longrightarrow $-$. This means that are produced in total $l \cdot Q \equiv (\#_-)_\Gamma$ changes. The same is true for the Λ submatrix but with $m \cdot Q \equiv (\#_-)_\Lambda$ changes the type $-$ \longrightarrow $+$. So we have

$$\text{ran}(V_+) \cdot Q + \text{ran}(V_-) \cdot Q = 2 \frac{\text{ran}(\mathcal{G})}{2} \cdot Q \quad (48)$$

$$= \text{ran}(\mathcal{G}) \cdot Q = 2n \cdot Q = \#_- \quad (49)$$

changes of sign.

The important thing here is how this change is distributed in ${}_d(X, X)_{S-\text{exp}}$. Algebras set to I , the amount of negative elements equals the number of positive elements. So the difference between these amounts (difference between the number of positive and negative diagonal elements without regard to the numerical value) is given by $\chi = \lambda_+ - \lambda_- = 0 = \chi_{S-\text{exp}}$.

It should be noted that although $\#_-$ is an even number important thing is how the change of sign is distributed in the diagonal elements of ${}_d(X, X)_{S-\text{exp}}$. This distribution generates changes in the number χ and plays an important role in the general classification of real forms of a complex Lie algebra.

1. *Case of the sets II and III*

In these cases,

$${}_d(g_{AB})_I = \begin{pmatrix} \Gamma & O \\ O & \Lambda \end{pmatrix}$$

where

$$\begin{aligned}
d(g_{AB})_{\Gamma} &= \begin{pmatrix} \lambda_{1d}(M_K) & & & & \\ & \ddots & & & \\ & & \lambda_{id}(M_K) & & \\ & & & \lambda_{i+1}(\bar{\lambda}_{Q+1}) & \\ & O & & & \ddots \\ & & & & & \lambda_{l \neq m}(\bar{\lambda}_P) \end{pmatrix}_{\substack{(l \cdot P) \times (l \cdot P) \\ \neq \\ (m \cdot P) \times (m \cdot P)}} \\
d(g_{AB})_{\Lambda} &= \begin{pmatrix} -\lambda_{l+1} d(M_K) & & & & \\ & \ddots & & & \\ & & -\lambda_{l+i} d(M_K) & & \\ & & & -\lambda_{l+i+1} d(\bar{\lambda}_{Q+1}) & \\ & O & & & \ddots \\ & & & & & -\lambda_{l+m} d(M_K) \end{pmatrix}_{\substack{(m \cdot P) \times (m \cdot P) \\ \neq \\ (l \cdot P) \times (l \cdot P)}}
\end{aligned}$$

where

$$d(M_K) = \begin{pmatrix} -\bar{\lambda}_1 & & & & \\ & \ddots & & & \\ & & -\bar{\lambda}_Q & & \\ & & & \bar{\lambda}_{Q+1} & \\ O & & & & \ddots \\ & & & & & \bar{\lambda}_P \end{pmatrix}$$

in this case occur $l \cdot Q = (\#_-)_{\Gamma}$ changes of sign $+\longrightarrow -$, and $m \cdot Q = (\#_-)_{\Lambda}$ changes of sign $-\longrightarrow +$, in a total of

$$\begin{aligned}
\#_{\Gamma} + \#_{\Lambda} &= \text{ran}(V_+) \cdot Q + \text{ran}(V_-) \cdot Q \\
&= l \cdot Q + m \cdot Q \\
&= \text{ran}(\mathcal{G}) \cdot Q = 2n \cdot Q = \#_-
\end{aligned}$$

where we can see that the value $\#$ is similar to the case of the set I , for the same value of $n \in \mathbb{N}$. However the outcome of interest in this case is

$$\begin{aligned}
l \cdot Q &\neq m \cdot Q \\
&\implies \\
(\#_-)_{\Gamma} &\neq (\#_-)_{\Lambda} \\
&\implies \\
\chi_I &= (\lambda_+ - \lambda_-)_I \neq (\lambda_+ - \lambda_-)_{II} = \chi_{II} .
\end{aligned} \tag{50}$$

This is due to the fact that the transformation in the signature of $_d(X, X)_{S-\exp}$ appears differently in each subspace, when the original algebra is of type *II* and *III*.

Consider now the relationship between the elements of the set $\{ran(V_+), ran(V_-), P, Q\}$ during the process of change in the value of χ difference. To find this relationship let's see what happens during the process:

- Increase of rank of V_+

$$ran(V_+) \longrightarrow ran(V_+) \cdot P$$

- Reduction in the rank under the change of internal sign

$$ran(V_+) \cdot P \longrightarrow ran(V_+) \cdot P - (\#_-)_{\Gamma}$$

- Increase of the rank due to internal change signature of another subspace

$$ran(V_+) \cdot P - (\#_-)_{\Gamma} \longrightarrow ran(V_+) \cdot P - (\#_-)_{\Gamma} + (\#_-)_{\Lambda} ,$$

that rewritten in terms of the elements of the set $\{ran(V_+), ran(V_-), P, Q\}$ we have (considering semisimplicity)

$$\begin{aligned}
ran(V_+) \cdot P - (\#_-)_{\Gamma} + (\#_-)_{\Lambda} &= ran(V_+) \cdot P - ran(V_+) \cdot Q + ran(V_-) \cdot Q \\
&= ran(V_+) \cdot P - \chi \cdot Q
\end{aligned}$$

So the character of algebra $\chi_{S-\exp}$ is given by

$$\begin{aligned}
\chi_{S-\text{exp}} &= \text{ran}(V_+)_{S-\text{exp}} - \text{ran}(V_-)_{S-\text{exp}} \\
&= \text{ran}(V_+) \cdot P - \text{ran}(V_+) \cdot Q + \text{ran}(V_-) \cdot Q \\
&\quad - (\text{ran}(V_-) \cdot P - \text{ran}(V_-) \cdot Q + \text{ran}(V_+) \cdot Q) \\
&= (\text{ran}(V_+) - \text{ran}(V_-)) \cdot (P - 2Q) \\
&= \chi \cdot (P - 2Q).
\end{aligned} \tag{51}$$

For nilpotent subspaces if they exist we have the following additional process

- Increase in the dimensionality of V_0

$$V_0 \longrightarrow V_0 \cdot P$$

- Increase in the dimensionality because the occurrence of zeros in V_{\pm} .

$$V_0 \longrightarrow V_0 \cdot p \longrightarrow V_0 \cdot P + (V_+ + V_-) \cdot H \tag{52}$$

where "H" denotes the increase.

From this we can see that for Lie algebras of the set I have

$$\chi_{S-\text{exp}} = \chi \cdot (P - 2Q) = 0$$

that is, regardless of the order semigroup or the amount of eigenvalues negative or zero of the matrix M_K the value of the difference $\chi_{S-\text{exp}}$ will always be zero. So, we have that algebras of type I will keep type I.

For the sets II , III is necessary to consider the following points

- The set of natural numbers (including zero) forms a group under addition but it is not group under the operation of subtraction: That is, it is not necessarily true that

$$\{a, b\} \in \mathbb{N}^* / a - b \in \mathbb{N}^*$$

- The set of even numbers form a group under addition and subtraction.
- The set of odd numbers do not form a group under subtraction nor under addition.

- An even number plus an odd number is always odd.

This means that the sets II and III satisfy the following properties:

$$\begin{aligned} \underbrace{ran(V_+)}_{\in \{2\mathbb{Z}+1\}} + \underbrace{ran(V_-)}_{\in \{2\mathbb{Z}+1\}} &\in II \\ \underbrace{ran(V_+)}_{\in \{2\mathbb{Z}\}} + \underbrace{ran(V_-)}_{\in \{2\mathbb{Z}\}} &\in II \end{aligned} \quad (53)$$

$$\begin{aligned} \underbrace{ran(V_+)}_{\in \{2\mathbb{Z}\}} + \underbrace{ran(V_-)}_{\in \{2\mathbb{Z}+1\}} &\in III \\ \underbrace{ran(V_+)}_{\in \{2\mathbb{Z}+1\}} + \underbrace{ran(V_-)}_{\in \{2\mathbb{Z}\}} &\in III \end{aligned} \quad (54)$$

and then, independently of the semigroup, we have

$$\begin{aligned} \chi_{S-\text{exp}} &= \chi \cdot (P - 2Q) \xrightarrow{I} \chi_{S-\text{exp}} = 0 \cdot (P - 2Q) \\ \chi_{S-\text{exp}} &= \chi \cdot (P - 2Q) \xrightarrow{II} \chi_{S-\text{exp}} = (2m - 2n) \cdot (P - 2Q) \\ \chi_{S-\text{exp}} &= \chi \cdot (P - 2Q) \xrightarrow{III} \chi_{S-\text{exp}} = \begin{cases} [2m + 1 - 2n] \cdot (P - 2Q) \\ [2m - 2n - 1] \cdot (P - 2Q) \end{cases} \\ \text{con } m, n &\in \mathbb{N} \end{aligned}$$

when $ran(V_{\pm}) \neq 0$.

This shows that $\chi_{S-\text{exp}}$ depends both the original χ value and semigroup characteristics.

An interesting question is: when it will produce a $\chi_{S-\text{exp}}$ with different sign to the original χ ?. To answer consider the following results:

- If

$$(\mp) \chi_{S-\text{exp}} = (\pm) \chi \cdot (P - 2Q),$$

that is

$$P - 2Q < 0, \quad (55)$$

then

$$\frac{P}{2} < Q .$$

2) If

$$(\pm) \chi_{S-\text{exp}} = (\pm) \chi \cdot (P - 2Q) ,$$

that is

$$P - 2Q > 0 \tag{56}$$

then

$$\frac{P}{2} > Q .$$

In other words, when $\text{ran}(V_{\pm}) \neq 0$ and the matrix has a number Q of negative eigenvalues greater than half of the quantity P (semigroup elements), a change will occur in the signature of the type $(\pm) \chi \longrightarrow (\mp) \chi_{S-\text{exp}}$ independently of the original algebra.

Furthermore, when the M_K matrix has a Q number of negative eigenvalues smaller than half of the of P (we must remember that the condition $P/2 \in \mathbb{N}$ has to be satisfied.), then there will be a process of the type $(\pm) \chi \longrightarrow (\pm) \chi_{S-\text{exp}}$ independently of the original algebra.

Consider now the study of the intrinsic geometry for the case that M_K has null eigenvalues $\bar{\lambda} = 0$.

Consider the following process

- Increasing the V_+ dimensionality of, for example,

$$\text{ran}(V_+) \longrightarrow \text{ran}(V_+) \cdot P$$

- Decreased dimensionality due to the appearance of internal zeros

$$\text{ran}(V_+) \cdot P \longrightarrow \text{ran}(V_+) \cdot P - \text{ran}(V_+) \cdot H = \text{ran}(V_+) \cdot (P - H)$$

- The dimensionality of V_+ , is not affected by the occurrence of zero elements in the complementary space, in this case V_-

$$\text{ran}(V_+) \cdot P - \text{ran}(V_+) \cdot H \longrightarrow \text{ran}(V_+) \cdot P - \text{ran}(V_+) \cdot H$$

thus $\chi_{S-\text{exp}}$ in the case $\bar{\lambda} = 0$, is given by

$$\begin{aligned}
\chi_{S-\text{exp}} &= \text{ran}(V_+)_{S-\text{exp}} - \text{ran}(V_-)_{S-\text{exp}} \\
&= \text{ran}(V_+) \cdot P - \text{ran}(V_+) \cdot H - [\text{ran}(V_-) \cdot P - \text{ran}(V_-) \cdot H] \\
&= \chi \cdot (P - H)
\end{aligned} \tag{57}$$

for $H \leq P$ always.

In this case it is not possible a process of the type $(\pm) \chi \longrightarrow (\mp) \chi_{S-\text{exp}}$ because the condition $H \leq P$, impedes that $\chi \cdot (P - H)$ having a different sign from the sign of χ .

We should also note that if a matrix has eigenvalues null then a decrease in their range occurs. This can be seen from the fact that

$$\text{ran}(\mathcal{G})_{S-\text{exp}} = d \cdot P \tag{58}$$

where "d" denotes the number of non-zero diagonal elements of $_d(X, X)$. The amount $(\text{ran}(\mathcal{G})_{S-\text{exp}})$ decrease if the rank of the matrix M_K decreases. Since there are $N = \dim(\mathcal{G})$ matrices M_K in $(X, X)_{S-\text{exp}}$, we have that for each eigenvalue zero of M_K matrices, there will be a decrease in the total range given by

$$\begin{aligned}
\text{decrease of the rank} &= H \cdot N \\
&\text{with } H \leq P
\end{aligned} \tag{59}$$

$$\begin{aligned}
\text{ran}\left(_d(X, X)_{S-\text{exp}}\right) &= \text{ran}(_d(X, X)) \cdot P - _d(X, X) \cdot H \\
&= \text{ran}(_d(X, X)) \cdot (P - H) .
\end{aligned}$$

For $\dim(V_0) = 0$, we have

$$\text{ran}(\mathcal{G})_{S-\text{exp}} = \text{ran}(\mathcal{G})(P - H)$$

which allows to calculate the expression for the change in the amount $\chi_{S-\text{exp}}$ in the presence of positive, negative and zero eigenvalues ($\dim(V_{\pm}) \neq 0$)

$$\begin{aligned}
\chi_{S\text{-exp}} &= \text{ran}(V_+) \cdot P - \text{ran}(V_+) \cdot Q - \text{ran}(V_+) \cdot H + \text{ran}(V_-) \cdot Q \\
&\quad - [\text{ran}(V_-) \cdot P - \text{ran}(V_-) \cdot Q - \text{ran}(V_-) \cdot H + \text{ran}(V_+) \cdot Q] \\
&= \chi(P - H - 2Q).
\end{aligned} \tag{60}$$

Let n_+, s_+, N_+ denote the number of positive eigenvalues, n_-, s_-, N_- the number of negative eigenvalues and n_0, s_0, N_0 the number of zero eigenvalues (all eigenvalues are counted with multiplicity). That is, with respect to the notation used: $P = s_+ + s_- + s_0$; $Q = s_-$ and $H = s_0$; Moreover $\chi_{S\text{-exp}} := s_+ - s_-$. The discussion about the signature of the S-expansion given above, could be resumed by the following theorem, in terms of the signatures of the original algebra and of the matrix M associated to the semigroup.

Theorem: Let \mathcal{G} be a real Lie algebra of dimension n whose Killing form has signature (n_+, n_-, n_0) and let S a semigroup of order s whose associated matrix M_k has signature (s_+, s_-, s_0) . Then the Killing form of the S -expanded algebra $\mathcal{G}_{S\text{-exp}}$ is (N_+, N_-, N_0) , where

$$N_+ = n_+ s_+ + n_- s_- \quad , \quad N_- = n_- s_+ + n_+ s_- \quad , \quad N_0 = n s_0 + s n_0. \tag{61}$$

In particular, the rank of the form for the S -expansion is

$$\text{rank}(S \times \mathcal{G}) = N_+ + N_- = (n_+ + n_-)(s_+ + s_-) = \text{rank}(M_k) \text{rank}(\mathcal{G}) \tag{62}$$

and

$$\chi_{(S \times \mathcal{G})} = N_+ - N_- = (n_+ - n_-)(s_+ - s_-) = \chi_{M_k} \chi_{\mathcal{G}}$$

Proof. This follows from the diagonalized form of the matrix for the Killing form of the S -expanded Lie algebra giving at the begining of this section for the matrix associated to de equation (35), wich shows that the eigenvalues are of the form $\lambda_i \bar{\lambda}_j$, where λ_i is an eigenvalue for \mathcal{G} and $\bar{\lambda}_j$ an eigenvalue for M_k . This implies that a positive eigenvalue is either the product of two positive or two negative eigenvalues. A negative eigenvalue is the product of a negative and a positive eigenvalue, or viceversa. Finally the zero eigenvalue is obtained as the product of a zero eigenvalue with any other eigenvalue.

Example: Consider the following simple example. Using $\chi = -1$, $\text{ran}(V_+) = 1$, $\text{ran}(V_-) = 2$, $P = 4$, $Q = 3$, we find that the expanded metric takes the form

$$d(X, X)_{S-\text{exp}} = \begin{pmatrix} D_+ & O \\ O & D_- \end{pmatrix}$$

where

$$D_{\pm} = \pm \lambda (\pm \bar{\lambda}) \cdot (I)_{\text{ran}(V_{\pm}) \cdot P \times \text{ran}(V_{\pm}) \cdot P} .$$

Since $P = 4$ and $Q = 3$, we can write

$$\begin{aligned} D_+ &= \begin{pmatrix} +\lambda(+\bar{\lambda}) & & & O \\ & +\lambda(-\bar{\lambda}) & & \\ & & +\lambda(-\bar{\lambda}) & \\ O & & & +\lambda(-\bar{\lambda}) \end{pmatrix} = \begin{pmatrix} +\lambda(+\bar{\lambda}) & & & O \\ & -\lambda\bar{\lambda} & & \\ & & -\lambda\bar{\lambda} & \\ O & & & -\lambda\bar{\lambda} \end{pmatrix} \\ D_- &= \begin{pmatrix} -\lambda(+\bar{\lambda}) & & & O \\ & -\lambda(-\bar{\lambda}) & & \\ & & -\lambda(-\bar{\lambda}) & \\ O & & & -\lambda(-\bar{\lambda}) \end{pmatrix} = \begin{pmatrix} -\lambda(+\bar{\lambda}) & & & O \\ & +\lambda\bar{\lambda} & & \\ & & +\lambda\bar{\lambda} & \\ O & & & +\lambda\bar{\lambda} \end{pmatrix} \end{aligned}$$

which leads to the following change in the dimensionality of the subspaces

$$\text{ran}(V_+)_{S-\text{exp}} = 1 \cdot P - 1 \cdot Q + 2 \cdot Q = 1 \cdot 4 - 1 \cdot 3 + 2 \cdot 3 = 7$$

$$\text{ran}(V_-)_{S-\text{exp}} = 2 \cdot P - 2 \cdot Q + 1 \cdot Q = 2 \cdot 4 - 2 \cdot 3 + 1 \cdot 3 = 5$$

so that

$$\chi_{S-\text{exp}} = \text{ran}(V_+)_{S-\text{exp}} - \text{ran}(V_-)_{S-\text{exp}} = 7 - 5 = 2 .$$

Thus

$$\chi_{S-\text{exp}} = \chi \cdot (P - 2Q) = -1 \cdot (4 - 6) = -1 \cdot -2 = 2 .$$

In the case that $P = 4, Q = 2, H = 1$, we have

$$D_+ = \begin{pmatrix} +\lambda(+\bar{\lambda}) & & & O \\ & +\lambda(-\bar{\lambda}) & & \\ & & +\lambda(-\bar{\lambda}) & \\ O & & & +\lambda(0) \end{pmatrix} = \begin{pmatrix} +\lambda(+\bar{\lambda}) & & & O \\ & -\lambda\bar{\lambda} & & \\ & & -\lambda\bar{\lambda} & \\ O & & & 0 \end{pmatrix}$$

$$D_- = \begin{pmatrix} -\lambda(+\bar{\lambda}) & & & O \\ & -\lambda(-\bar{\lambda}) & & \\ & & -\lambda(-\bar{\lambda}) & \\ O & & & -\lambda(0) \end{pmatrix} = \begin{pmatrix} -\lambda(+\bar{\lambda}) & & & O \\ & +\lambda\bar{\lambda} & & \\ & & +\lambda\bar{\lambda} & \\ O & & & 0 \end{pmatrix}$$

from where

$$\begin{aligned} \text{ran}(V_+)_{S-\text{exp}} &= \text{ran}(V_+) \cdot P - \text{ran}(V_+) \cdot H - \text{ran}(V_+) \cdot Q + \text{ran}(V_-) \cdot Q \\ &= 1 \cdot 4 - 1 \cdot 1 - 1 \cdot 2 + 2 \cdot 2 = 5 \end{aligned}$$

$$\begin{aligned} \text{ran}(V_-)_{S-\text{exp}} &= \text{ran}(V_-) \cdot P - \text{ran}(V_-) \cdot H - \text{ran}(V_-) \cdot Q + \text{ran}(V_+) \cdot Q \\ &= 2 \cdot 4 - 2 \cdot 1 - 2 \cdot 2 + 1 \cdot 2 = 4 \end{aligned}$$

and furthermore

$$\chi_{S-\text{exp}} = \text{ran}(V_+)_{S-\text{exp}} - \text{ran}(V_-)_{S-\text{exp}} = 5 - 1 = 1 .$$

On the other hand, using the equation for $\chi_{S-\text{exp}}$ in the presence of $\bar{\lambda} = 0$ elements, it is found

$$\chi_{S-\text{exp}} = \chi \cdot (P - H - 2Q) = -1 \cdot (4 - 1 - 4) = 1$$

These results establish the conditions to enable two Lie algebras can be obtained one from the other, by the S -expansion procedure.

If there are no negative or null eigenvalues, we will have

$$\begin{aligned} \chi_{S-\text{exp}} &= \text{ran}(V_+) \cdot P - \text{ran}(V_-) \cdot P \\ &= \chi \cdot P , \end{aligned}$$

on the other hand,

$$\begin{aligned}\chi_{S-\text{exp}} &= \chi \cdot (P - H - 2Q) \\ \text{If } H = Q = 0, \quad \chi_{S-\text{exp}} &= \chi \cdot P\end{aligned}$$

B. Characteristics of the term $K_{i\gamma}^\delta K_{j\delta}^\gamma$

Since $K_{\alpha\beta}^\gamma \in \{0, 1\}$ we have $K_{i\gamma}^\delta K_{j\delta}^\gamma \in \{0, 1, 2, \dots, P\}$ ($\#$ denoting cardinality) with indices i and j in the set $\{1, 2, \dots, P\}$. This statement can be corroborated by the following example: Consider the product $K_{i\gamma}^\delta K_{j\delta}^\gamma$ for the case $i, j = 1, 1$:

$$\begin{aligned}K_{1\gamma}^\delta K_{1\delta}^\gamma &= K_{11}^\delta K_{1\delta}^1 + K_{12}^\delta K_{1\delta}^2 + \dots + K_{1P}^\delta K_{1\delta}^P \\ &= K_{11}^1 K_{11}^1 + K_{11}^2 K_{12}^1 + \dots + K_{11}^P K_{1P}^1 \\ &+ K_{12}^1 K_{11}^2 + K_{12}^2 K_{12}^2 + \dots + K_{12}^P K_{1P}^2 \\ &\vdots \\ &+ K_{1P}^1 K_{11}^P + K_{13}^2 K_{12}^3 + \dots + K_{1P}^P K_{1P}^P.\end{aligned}\tag{63}$$

Note that not all summands may be different from zero. If this happens could not be univocally defined the composition law semigroup. For example if we assume that

$$K_{13}^1 K_{11}^3 \neq 0 \ ; \ K_{13}^2 K_{12}^3 \neq 0 \ ; \ \dots \ K_{13}^P K_{1P}^3 \neq 0\tag{64}$$

we would come to a contradiction. Indeed, if $K_{13}^1 K_{11}^3 \neq 0$ then $K_{13}^1 \neq 0 \ \wedge \ K_{11}^3 \neq 0$, so that the products $s_1 \diamond s_3$ and $s_1 \diamond s_1$ are defined as $s_1 \diamond s_3 = s_1$ and $s_1 \diamond s_1 = s_3$. This means that due to this choice $K_{13}^2 K_{12}^3 = 0, \dots, K_{13}^P K_{1P}^3 = 0$.

Since this is true for any pair of indices i, j we have that the maximum value of $K_{i\gamma}^\delta K_{j\delta}^\gamma$ is equal to the number of elements that form the semigroup. So we have:

$$\begin{aligned}
K_{1\gamma}^\delta K_{1\delta}^\gamma &= K_{11}^\delta K_{1\delta}^1 + K_{12}^\delta K_{1\delta}^2 + K_{13}^\delta K_{1\delta}^3 \\
&= K_{11}^1 K_{11}^1 + K_{11}^2 K_{12}^1 + \cdots + K_{11}^P K_{1P}^1 \\
&\quad + K_{12}^1 K_{11}^2 + K_{12}^2 K_{12}^2 + \cdots + K_{12}^P K_{1P}^2 \\
&\quad \vdots \\
&\quad + K_{13}^1 K_{11}^3 + K_{13}^2 K_{12}^3 + \cdots + K_{1P}^P K_{1P}^P \\
&= 1 + 1 + \cdots + 1 = P \quad \forall \{i, j\} \in \{1, 2, \dots, P\}
\end{aligned}$$

This has the consequence that all KK pairs that appear in the expanded metric multiplied by the coefficients associated with the original algebra, have at most a magnitude equal to "order" P of the semigroup.

It should be noted that if the semigroup has P elements, then there must exist a "z" number of K -selectors that satisfy the condition $K_{i\gamma}^\delta \neq 0$ for some combination of their indices, according to the following table:

$$\begin{aligned}
P_1 &= 1 \\
z &= 1 \\
P_2 &= 2 \\
z &= P_1 + P_2 = 1 + 2 \\
P_3 &= 3 \\
z &= P_1 + P_2 + P_3 = 1 + 2 + 3 \\
&\vdots \\
P_P &= P \\
z &= p_1 + p_2 + p_3 + \cdots + p_p = 1 + 2 + 3 + \cdots + P \\
&= \sum_{i=1}^P P_i = \sum_{i=1}^P i = \frac{i(i+1)}{2}
\end{aligned} \tag{65}$$

it's interesting that this is just the Faulhaber's formula $\sum_{i=1}^P i^n$ for $n = 1$. That expresses the well known "triangular numbers" showed in ref. [22]

C. Vector Magnitude

An interesting consequence of the above property is that $K_{\alpha\gamma}^\delta K_{\alpha\delta}^\gamma \in \{0, 1, 2, \dots, P\}$ affects the measurement of the length of the original basis vectors. Indeed,

$$\begin{aligned}
\|X_\Phi\| &= \sqrt{(X_\Phi, X_\Phi)_{S-\text{exp}}} = \sqrt{\text{tr}(R(X_\Phi)R(X_\Phi))} \\
&= \sqrt{R(X_\Phi)_\Omega^\Theta R(X_\Phi)_\Theta^\Omega} = \sqrt{(C_\Phi)_\Omega^\Theta (C_\Phi)_\Theta^\Omega} = \sqrt{(C_{(\alpha,A)})_{(\gamma,C)}^{(\delta,D)} (C_{(\alpha,A)})_{(\delta,D)}^{(\gamma,C)}} \\
&= \sqrt{K_{\alpha\gamma}^\delta K_{\alpha\delta}^\gamma C_{AC}^D C_{AD}^C} = \sqrt{K_{\alpha\gamma}^\delta K_{\alpha\delta}^\gamma} \sqrt{C_{AC}^D C_{AD}^C} = \sqrt{K_{\alpha\gamma}^\delta K_{\alpha\delta}^\gamma} \sqrt{g_{AA}} = \sqrt{K_{\alpha\gamma}^\delta K_{\alpha\delta}^\gamma} \|X_A\|
\end{aligned} \tag{66}$$

$$\begin{aligned}
\|X_{\Phi=(\alpha,A)}\| &= \sqrt{K_{\alpha\gamma}^\delta K_{\alpha\delta}^\gamma} \|X_A\| \\
&\in \left\{ \begin{array}{l} \sqrt{0 \cdot g_{AA}} \equiv \sqrt{0} \|X_A\| = g_{(\alpha,A)(\alpha,A)} \\ \sqrt{1 \cdot g_{AA}} \equiv \sqrt{1} \|X_A\| = g_{(\alpha,A)(\alpha,A)} \\ \sqrt{2 \cdot g_{AA}} \equiv \sqrt{2} \|X_A\| = g_{(\alpha,A)(\alpha,A)} \\ \sqrt{3 \cdot g_{AA}} \equiv \sqrt{3} \|X_A\| = g_{(\alpha,A)(\alpha,A)} \\ \vdots \\ \sqrt{P \cdot g_{AA}} \equiv \sqrt{P} \|X_A\| = g_{(\alpha,A)(\alpha,A)} \end{array} \right\}
\end{aligned} \tag{67}$$

or in the general case

$$\|V\| = \sqrt{(v^{(\alpha,A)})^2 g_{(\alpha,A)(\alpha,A)}} = v^{(\alpha,A)} \sqrt{K_{\alpha\gamma}^\delta K_{\alpha\delta}^\gamma} \|X_A\| \tag{68}$$

This means that the metric tensor components experience a rescaling. We can also see that the rescaling of vectors depends on elements of $K_{i\gamma}^\delta K_{j\delta}^\gamma \longrightarrow K_{\alpha\gamma}^\delta K_{\alpha\delta}^\gamma$, present in the diagonal of the expanded metric tensor.

D. Angular spacing between vectors

The S-expansion procedure affects the angle between two vectors in the $V_+ + V_- = (S \otimes \mathcal{G})/V_0 \subseteq S \otimes \mathcal{G}$ space. Indeed,

$$\begin{aligned}
\cos(\theta)_{S-\exp} &= \frac{(X_\Phi, X_\Theta)_{S-\exp}}{\sqrt{(X_\Phi, X_\Phi)_{S-\exp}} \sqrt{(X_\Theta, X_\Theta)_{S-\exp}}} \equiv \frac{(X_\Phi, X_\Theta)}{\sqrt{(X_\Phi, X_\Phi)} \sqrt{(X_\Theta, X_\Theta)}} \\
&= \frac{\text{tr}(R(X_\Phi) R(X_\Theta))}{\sqrt{\text{tr}(R(X_\Phi) R(X_\Phi))} \sqrt{\text{tr}(R(X_\Theta) R(X_\Theta))}} \\
&= \frac{R(X_\Phi)_\Omega^\Gamma R(X_\Theta)_\Gamma^\Omega}{\sqrt{\text{tr} R(X_\Phi)_\Omega^\Gamma R(X_\Phi)_\Gamma^\Omega} \sqrt{\text{tr} R(X_\Theta)_\Omega^\Gamma R(X_\Theta)_\Gamma^\Omega}} \tag{69}
\end{aligned}$$

$$\begin{aligned}
&= \frac{(C_\Phi)_\Omega^\Gamma (C_\Theta)_\Gamma^\Omega}{\sqrt{(C_\Phi)_\Omega^\Gamma (C_\Phi)_\Gamma^\Omega} \sqrt{(C_\Theta)_\Omega^\Gamma (C_\Theta)_\Gamma^\Omega}} \\
&= \frac{(C_{(\alpha,A)})_{(\gamma,C)}^{(\delta,D)} (C_{(\beta,B)})_{(\delta,D)}^{(\gamma,C)}}{\sqrt{(C_{(\alpha,A)})_{(\gamma,C)}^{(\delta,D)} (C_{(\alpha,A)})_{(\delta,D)}^{(\gamma,C)}} \sqrt{(C_{(\beta,B)})_{(\gamma,C)}^{(\delta,D)} (C_{(\beta,B)})_{(\delta,D)}^{(\gamma,C)}}} \\
&= \frac{K_{\alpha\gamma}^\delta K_{\beta\delta}^\gamma (C_A)_C^D (C_B)_D^C}{\sqrt{K_{\alpha\gamma}^\delta K_{\alpha\delta}^\gamma (C_A)_C^D (C_A)_D^C} \sqrt{K_{\beta\gamma}^\delta K_{\beta\delta}^\gamma (C_B)_C^D (C_B)_D^C}}
\end{aligned} \tag{70}$$

$$\begin{aligned}
&= \frac{K_{\alpha\gamma}^\delta K_{\beta\delta}^\gamma}{\sqrt{K_{\alpha\gamma}^\delta K_{\alpha\delta}^\gamma} \sqrt{K_{\beta\gamma}^\delta K_{\beta\delta}^\gamma}} \frac{(C_A)_C^D (C_B)_D^C}{\sqrt{(C_A)_C^D (C_A)_D^C} \sqrt{(C_B)_C^D (C_B)_D^C}} \\
(X_{(\alpha,A)}, X_{(\beta,B)}) &= \frac{K_{\alpha\gamma}^\delta K_{\beta\delta}^\gamma}{\sqrt{K_{\alpha\gamma}^\delta K_{\alpha\delta}^\gamma} \sqrt{K_{\beta\gamma}^\delta K_{\beta\delta}^\gamma}} \frac{g_{AB}}{\sqrt{g_{AA}} \sqrt{g_{BB}}} \\
&= \frac{\sum K_{\alpha\gamma}^\delta K_{\beta\delta}^\gamma}{\sqrt{\sum K_{\alpha\gamma}^\delta K_{\alpha\delta}^\gamma} \sqrt{\sum K_{\beta\gamma}^\delta K_{\beta\delta}^\gamma}} \cos(\theta) \tag{71}
\end{aligned}$$

In general we find that,

$$\begin{aligned}
(v^{(\alpha,A)} X_{(\alpha,A)}, v^{(\beta,B)} X_{(\beta,B)}) &= v^{(\alpha,A)} v^{(\beta,B)} \frac{K_{\alpha\gamma}^\delta K_{\beta\delta}^\gamma}{\sqrt{K_{\alpha\gamma}^\delta K_{\alpha\delta}^\gamma} \sqrt{K_{\beta\gamma}^\delta K_{\beta\delta}^\gamma}} \frac{g_{AB}}{\sqrt{g_{AA}} \sqrt{g_{BB}}} \\
&= v^{(\alpha,A)} v^{(\beta,B)} \frac{\sum K_{\alpha\gamma}^\delta K_{\beta\delta}^\gamma}{\sqrt{\sum K_{\alpha\gamma}^\delta K_{\alpha\delta}^\gamma} \sqrt{\sum K_{\beta\gamma}^\delta K_{\beta\delta}^\gamma}} \cos(\theta) \tag{72}
\end{aligned}$$

So that

$$\begin{aligned}\cos(\Theta) &= (X_{(\alpha,A)}, X_{(\beta,B)}) = \frac{\sum K_{\alpha\gamma}^\delta K_{\beta\delta}^\gamma}{\sqrt{K_{\alpha\gamma}^\delta K_{\alpha\delta}^\gamma} \sqrt{K_{\beta\gamma}^\delta K_{\beta\delta}^\gamma}} \frac{g_{AB}}{\sqrt{g_{AA}} \sqrt{g_{BB}}} \\ &= \frac{\sum K_{\alpha\gamma}^\delta K_{\beta\delta}^\gamma}{\sqrt{\sum K_{\alpha\gamma}^\delta K_{\alpha\delta}^\gamma} \sqrt{\sum K_{\beta\gamma}^\delta K_{\beta\delta}^\gamma}} \cos(\theta) = n \in \mathbb{R}\end{aligned}$$

These results lead to define a number denoted as Δ which depends on the composition law of the semigroup. In fact the number Δ is defined as

$$\Delta \equiv \left\{ \frac{[0, 1, 2, 3, \dots, P]_{i,j}}{\sqrt{[0, 1, 2, 3, \dots, P]_{i,i}} \sqrt{[0, 1, 2, 3, \dots, P]_{j,j}}} \right\} \in \mathbb{R}^+,$$

from where we can see that Δ depends on the composition law semigroup and that is a real number. These results also lead to an important condition for the diagonalization of the S -expanded metric depending on the internal law of the semigroup. In fact the metric will be diagonal if:

$$\sum K_{i\gamma}^\delta K_{j\delta}^\gamma = 0 \quad (73)$$

for $i \neq j$.

Although the amount KK is positive semidefinite, it is possible that there is a base change such that the diagonal elements $K_{i\gamma}^\delta K_{i\delta}^\gamma$ change of sign.

However this does not affect the good performance of the function $\cos(\Theta)$ in a diagonal base.

Since the function $\cos(\Theta)$ is bounded in a continuous interval $[-1, 1]$, we have some values for Δ that are prohibited. In fact for an arbitrary value of θ angle, i, j fixed we could have

$$\cos(\Theta) = (X_{(i,A)}, X_{(j,B)}) = \frac{\sum K_{i\gamma}^\delta K_{j\delta}^\gamma}{\sqrt{\sum K_{i\gamma}^\delta K_{i\delta}^\gamma} \sqrt{\sum K_{j\gamma}^\delta K_{j\delta}^\gamma}} \cos(\theta) \neq \frac{0}{0},$$

from where we can see that the following conditions

$$\forall K_{i\gamma}^\delta K_{j\delta}^\gamma = 0, \quad K_{i\gamma}^\delta K_{i\delta}^\gamma \cdot K_{j\gamma}^\delta K_{j\delta}^\gamma \neq 0 \quad (74)$$

$$\forall K_{i\gamma}^\delta K_{j\delta}^\gamma \neq 0, \quad K_{i\gamma}^\delta K_{i\delta}^\gamma \cdot K_{j\gamma}^\delta K_{j\delta}^\gamma \neq 0 \quad (75)$$

must be fulfilled.

That is, in every case of the configurations for the S –expanded metric must be met that $K_{i\gamma}^\delta K_{i\delta}^\gamma \neq 0$ in order to prevent that the angular separation between vectors of this diagonal basis is ill-defined when we make the product $S \otimes \mathcal{G}$.

Since the function $\cos(\Theta)$ between two different basis vectors of these must be zero and not $0/0$ or ∞ , it is necessary that the diagonal elements $K_{i\gamma}^\delta K_{i\delta}^\gamma$ are different from zero, so that the angular separation of the new basis vectors is well defined. For example for the case $P = 3$, it is found

$$\cos(\Theta) = (X_{(1,A)}, X_{(3,B)}) = \frac{\sum K_{1\gamma}^\delta K_{3\delta}^\gamma}{\sqrt{\sum K_{1\gamma}^\delta K_{1\delta}^\gamma} \sqrt{\sum K_{3\gamma}^\delta K_{3\delta}^\gamma}} \cos(\theta)_{A,B}$$

$$\forall \quad K_{1\gamma}^\delta K_{1\delta}^\gamma \cdot K_{3\gamma}^\delta K_{3\delta}^\gamma \neq 0. \quad (76)$$

If X_A and X_B are ortogonal in the original algebra and then $\sum K_{1\gamma}^\delta K_{3\delta}^\gamma \cdot \cos(\theta)_{A,B} = 0$ and we could have, for example

$$\frac{\sum K_{1\gamma}^\delta K_{3\delta}^\gamma}{\sqrt{\sum K_{1\gamma}^\delta K_{1\delta}^\gamma} \sqrt{\sum K_{3\gamma}^\delta K_{3\delta}^\gamma}} \cos(\theta)_{A,B} = 0$$

But if further the term $\sum K_{1\gamma}^\delta K_{1\delta}^\gamma$ or the term $\sum K_{3\gamma}^\delta K_{3\delta}^\gamma$ is zero we have a mathematical problem regardless the value of $\sum K_{1\gamma}^\delta K_{3\delta}^\gamma$, then it's always necessary a M_K matrix with the form

$$M_K \equiv \begin{pmatrix} K_{1\gamma}^\delta K_{1\delta}^\gamma \neq 0 & * & * \\ * & * & * \\ * & * & K_{3\gamma}^\delta K_{3\delta}^\gamma \neq 0 \end{pmatrix}. \quad (77)$$

The boxes containing the symbol $*$ may be zero simultaneously or separately, for some choice of semigroup.

For other cases it is found

$$M_K \equiv \begin{pmatrix} K_{1\gamma}^\delta K_{1\delta}^\gamma \neq 0 & * & * \\ * & K_{2\gamma}^\delta K_{2\delta}^\gamma \neq 0 & * \\ * & * & * \end{pmatrix} \quad (78)$$

$$M_K \equiv \begin{pmatrix} * & * & * \\ * & K_{2\gamma}^\delta K_{2\delta}^\gamma \neq 0 & * \\ * & * & K_{3\gamma}^\delta K_{3\delta}^\gamma \neq 0 \end{pmatrix} \quad (79)$$

In summary we can say that both the magnitude of the vectors and the angle between them are strongly affected by S -expansion process. In more precise way they are affected by the composition law of the semigroup codified in the K -selectors.

It should be noted that in the last calculation, both indices of the original algebra as semigroup indices i, j are fixed. This means that the change in the magnitude or the angle θ depends on the sum of γ, δ indices. So if we want to change these geometric properties is necessary to impose conditions on the semigroup from this sum.

IV. THE SEMIGROUP: EXAMPLE $\mathfrak{so}(4)$ FROM $\mathfrak{so}(3)$

So far we have studied the effects it produces, the process of expansion, on the geometry of the manifold of the original Lie group. In particular, we have considered the effects on the metric of the manifold of Lie group that lead us to the metric of a new Lie group. The result of this Study allows to determine the geometrical role of the semigroup and its composition law. In this section is outlined, via an example, a method for determining the semigroup, which would provide a Lie algebra from another. This problem was recently addressed, from a slightly different viewpoint, in Refs. [19], [20].

A. Geometrical considerations

Consider first the case in which M_K matrices have no negative or zero eigenvalues. This means that our attention will focus on the coefficient " P " with the law of composition " \diamond " semigroup.

In this case obtaining an algebra from another shall be subject to following conditions:

$$\text{ran}(V_{\pm}) \xrightarrow{S\text{-exp}} \text{ran}(V_{\pm}) \cdot P \quad (A.1)$$

$$\text{ran}(\mathcal{G}) \xrightarrow{S\text{-exp}} \text{ran}(\mathcal{G}) \cdot P \quad (B.1)$$

$$\chi \xrightarrow{S\text{-exp}} \chi \cdot P. \quad (C.1)$$

For clarity, consider the example of obtaining $so(4)$ from $so(3)$. In this case one has

$$\text{ran}(so(3)) = 3$$

$$\chi_{so(3)} = -3$$

$$\text{ran}(so(4)) = 6$$

$$\chi_{so(4)} = -6$$

$$\text{ran}(V_-)_{so(3)} = 3$$

$$\text{ran}(V_+)_{so(3)} = 0$$

$$\text{ran}(V_-)_{so(4)} = 6$$

$$\text{ran}(V_+)_{so(4)} = 0.$$

This information allows to determine the number of elements that must have the semi-group connecting such algebras.

The character of the expanded algebra is given, as we have seen, by

$$\begin{aligned} \chi_{S\text{-exp}} &= \text{ran}(V_+) \cdot (P - H - Q) + \text{ran}(V_-) \cdot Q - [\text{ran}(V_-) \cdot (P - H - Q) + \text{ran}(V_+) \cdot Q] \\ &= -\text{ran}(V_-) \cdot (P - H - Q) = -3(P - H - Q) \end{aligned}$$

where we see that

$$-6 = -3(P - 0 - 0)$$

$$2 = P - 0 - 0$$

$$2 = P$$

The ranks of $(V_-)_{S-\text{exp}}$ and $\mathcal{G}_{S-\text{exp}}$ are given by

$$\begin{aligned}
\text{ran}(V_-)_{S-\text{exp}} &= \text{ran}(V_-) \cdot (P - H - Q) + \text{ran}(V_+) \cdot Q \\
6 &= 3 \cdot (P - H - Q) \\
2 &= P - 0 - 0 \\
\text{ran}(\mathcal{G})_{S-\text{exp}} &= \text{ran}(\mathcal{G}) \cdot (P - 0 - 0) \\
2 &= P
\end{aligned}$$

This is because the character $so(n)$ is given by $\chi = -\text{ran}(g_{ab})_{so(n)}$, because all the generators are compact. So we have that the number of elements is characterized by $P = 2$, $H = Q = 0$ and denoted by $S_{P,H,Q} \longrightarrow S_{2,0,0}$, i.e., the semigroup may be a semigroup of two elements whose matrix M_K does not have eigenvalues zero and negative eigenvalues.

Now consider the case where there are negative eigenvalues and $\text{ran}(V_{\pm}) \neq 0$. In this case it is required

$$\text{ran}(V_{\pm}) \xrightarrow{S-\text{exp}} \text{ran}(V_{\pm}) \cdot (P - Q) + \text{ran}(V_{\mp}) \cdot Q \quad (\text{A.2})$$

$$\text{ran}(\mathcal{G}) \xrightarrow{S-\text{exp}} \text{ran}(\mathcal{G}) \cdot P \quad (\text{B.2})$$

$$\chi \xrightarrow{S-\text{exp}} \chi \cdot (P - 2Q) \quad (\text{C.2})$$

and for the more general case, $\text{rang}(M_K) < P$, and again $(\text{ran}(V_{\pm}) \neq 0)$

$$\text{ran}(V_{\pm}) \xrightarrow{S-\text{exp}} \text{ran}(V_{\pm}) \cdot (P - H - Q) + \text{ran}(V_{\mp}) \cdot Q \quad (\text{A.3})$$

$$\text{ran}(\mathcal{G}) \xrightarrow{S-\text{exp}} \text{ran}(\mathcal{G}) \cdot (P - H) \quad (\text{B.3})$$

$$\chi \xrightarrow{S-\text{exp}} \chi \cdot (P - H - 2Q) \quad (\text{C.3})$$

This ensures that two Lie algebras A and B could be related by S -expansion, if the total of the above conditions are satisfy.

It should be noted that the above equations are not all indepedientes, because the equation for character χ is constructed based on the other. So we have that the equations

$$\text{ran}(V_{\pm}) \xrightarrow{S\text{-exp}} \text{ran}(V_{\pm}) \cdot (P - H - Q) + \text{ran}(V_{\mp}) \cdot Q \quad (80)$$

$$\text{ran}(\mathcal{G}) \xrightarrow{S\text{-exp}} \text{ran}(\mathcal{G}) \cdot (P - H) \quad (81)$$

are independent. This means it is possible that two Lie algebras could be linked for more than a semigroup in the event that P, H, Q are nonzero and belong to \mathbb{N}^* .

Consider now the conditions that lead to determining the semigroup, i.e. to the determination of the elements and its composition law.

B. Conditions on the semigroup

We have established the conditions on the intrinsic geometry generating conditions on the semigroup. Equivalently, the conditions on the metric leads to a set of values for the different elements $K_{i\gamma}^{\delta} K_{j\delta}^{\gamma}$. To clarify the idea consider a semigroup of two elements having a matrix M_K given by

$$M_K = \begin{pmatrix} K_{1\gamma}^{\delta} K_{1\delta}^{\gamma} & K_{1\gamma}^{\delta} K_{2\delta}^{\gamma} \\ K_{2\gamma}^{\delta} K_{1\delta}^{\gamma} & K_{2\gamma}^{\delta} K_{2\delta}^{\gamma} \end{pmatrix} \equiv \begin{pmatrix} a & b \\ b & c \end{pmatrix} \quad (82)$$

whose eigenvalues are given by

$$\begin{aligned} \lambda_1 &= \frac{1}{2}a + \frac{1}{2}c + \frac{1}{2}\sqrt{a^2 - 2ac + 4b^2 + c^2} \\ &= \frac{1}{2} \left(K_{1\gamma}^{\delta} K_{1\delta}^{\gamma} + K_{2\gamma}^{\delta} K_{2\delta}^{\gamma} + \sqrt{(K_{1\gamma}^{\delta} K_{1\delta}^{\gamma})^2 - 2(K_{1\gamma}^{\delta} K_{1\delta}^{\gamma})(K_{2\gamma}^{\delta} K_{2\delta}^{\gamma}) + 4(K_{1\gamma}^{\delta} K_{2\delta}^{\gamma})^2 + (K_{2\gamma}^{\delta} K_{2\delta}^{\gamma})^2} \right) \\ \lambda_2 &= \frac{1}{2}a + \frac{1}{2}c - \frac{1}{2}\sqrt{a^2 - 2ac + 4b^2 + c^2} \\ &= \frac{1}{2} \left(K_{1\gamma}^{\delta} K_{1\delta}^{\gamma} + K_{2\gamma}^{\delta} K_{2\delta}^{\gamma} - \sqrt{(K_{1\gamma}^{\delta} K_{1\delta}^{\gamma})^2 - 2(K_{1\gamma}^{\delta} K_{1\delta}^{\gamma})(K_{2\gamma}^{\delta} K_{2\delta}^{\gamma}) + 4(K_{1\gamma}^{\delta} K_{2\delta}^{\gamma})^2 + (K_{2\gamma}^{\delta} K_{2\delta}^{\gamma})^2} \right) \end{aligned} \quad (83)$$

Since M_K is a symmetric matrix, its eigenvalues $\lambda_{1,2} \in \mathbb{R}$, and therefore

$$\begin{aligned} (K_{1\gamma}^{\delta} K_{1\delta}^{\gamma})^2 - 2(K_{1\gamma}^{\delta} K_{1\delta}^{\gamma})(K_{2\gamma}^{\delta} K_{2\delta}^{\gamma}) + 4(K_{1\gamma}^{\delta} K_{2\delta}^{\gamma})^2 + (K_{2\gamma}^{\delta} K_{2\delta}^{\gamma})^2 &\geq 0 \\ (K_{1\gamma}^{\delta} K_{1\delta}^{\gamma} - K_{2\gamma}^{\delta} K_{2\delta}^{\gamma})^2 + 4(K_{1\gamma}^{\delta} K_{2\delta}^{\gamma})^2 &\geq 0. \end{aligned}$$

Since λ_2 can accept negative or zero values we have

$$\frac{1}{2}a + \frac{1}{2}c - \frac{1}{2}\sqrt{a^2 - 2ac + 4b^2 + c^2} \leq 0$$

$$ac \leq b^2$$

$$(K_{1\gamma}^\delta K_{1\delta}^\gamma) (K_{2\gamma}^\delta K_{2\delta}^\gamma) \leq (K_{1\gamma}^\delta K_{2\delta}^\gamma)^2 .$$

This result allows to modify the $\lambda_{1,2}$ values according to relationships between $K_{i\gamma}^\delta K_{j\delta}^\gamma$. Since: (i) each $K_{i\gamma}^\delta K_{j\delta}^\gamma$ object takes values ranging between 0 and P (or between 1 and P), which in this case are bounded by $\{0, 1, 2\}$ and (ii) the angular separation (orthogonality metric if diagonal) must be well defined in space $S \times \mathcal{G}$ irrespective of the chosen base; it is necessary that all elements of type $K_{i\gamma}^\delta K_{i\delta}^\gamma$ are nonzero. So we have to $\lambda_2 < 0$

$$(K_{1\gamma}^\delta K_{1\delta}^\gamma) (K_{2\gamma}^\delta K_{2\delta}^\gamma) < (K_{1\gamma}^\delta K_{2\delta}^\gamma)^2$$

$$1 \cdot 1 < 2^2 = 4$$

$$1 \cdot 2 < 2^2 = 4$$

$$2 \cdot 1 < 2^2 = 4$$

for the case of a null eigenvector:

$$(K_{1\gamma}^\delta K_{1\delta}^\gamma) (K_{2\gamma}^\delta K_{2\delta}^\gamma) = (K_{1\gamma}^\delta K_{2\delta}^\gamma)^2$$

$$1 \cdot 1 = 1^2 = 1$$

or

$$2 \cdot 2 = 2^2 = 4$$

Note that this condition is equivalent to $\det(M_K) = 0$. For a semigroup of order three M_K have the form

$$\begin{pmatrix} K_{1\gamma}^\delta K_{1\delta}^\gamma & K_{1\gamma}^\delta K_{2\delta}^\gamma & K_{1\gamma}^\delta K_{3\delta}^\gamma \\ K_{2\gamma}^\delta K_{1\delta}^\gamma & K_{2\gamma}^\delta K_{2\delta}^\gamma & K_{2\gamma}^\delta K_{3\delta}^\gamma \\ K_{3\gamma}^\delta K_{1\delta}^\gamma & K_{3\gamma}^\delta K_{2\delta}^\gamma & K_{3\gamma}^\delta K_{3\delta}^\gamma \end{pmatrix} \equiv \begin{pmatrix} a & b & c \\ b & d & e \\ c & e & f \end{pmatrix}$$

whose characteristic polynomial is

$$P(X) = X^3 + (-a - d - f)X^2 + (-b^2 - c^2 + f(a + d) + ad - e^2)X + (2ebc - fb^2 - dc^2 - ae^2 + adf) = 0. \quad (84)$$

So that

a) For $H = 1$, we find

$$-fb^2 + 2ebc - dc^2 - ae^2 + adf = 0 \quad (85)$$

therefore there is at least a null root for the polynomial $P(X)$.

b) For $H = 2$, we have

$$-fb^2 + 2ebc - dc^2 - ae^2 + adf = 0 \quad (I)$$

$$-b^2 - c^2 + f(a + d) + ad - e^2 = 0 \quad (II)$$

which produces at least two null roots in polynomial $P(X)$.

3) For $H = 3$

$$-fb^2 + 2ebc - dc^2 - ae^2 + adf = 0 \quad (I)$$

$$-b^2 - c^2 + f(a + d) + ad - e^2 = 0 \quad (II)$$

$$a + d + f = 0 \quad (III)$$

which produces at least three null roots in polynomial $P(X)$.

This allows to obtain conditions on $K_{i\gamma}^\delta K_{j\delta}^\gamma$ objects so that the semigroup and its composition law appear as a natural consequence of the loss of semisimplicity of the expanded algebra to $H \neq 0$

Let us see how determine the composition law semigroup $S_{P,H,Q}$.

C. Case of a semigroup of order 2

Consider an abelian finite and arbitrary semigroup of two elements

$$\begin{array}{ccc}
\Diamond & \lambda_1 & \lambda_2 \\
\lambda_1 & * & * \\
\lambda_2 & * & *
\end{array} . \tag{86}$$

If $Q = 1$ we find

$$(K_{1\gamma}^\delta K_{1\delta}^\gamma) (K_{2\gamma}^\delta K_{2\delta}^\gamma) < (K_{1\gamma}^\delta K_{2\delta}^\gamma)^2 .$$

In fact

$$K_{i\gamma}^\delta K_{j\delta}^\gamma = \{K_{1\gamma}^\delta K_{1\delta}^\gamma, K_{1\gamma}^\delta K_{2\delta}^\gamma, K_{2\gamma}^\delta K_{2\delta}^\gamma\}$$

$$\begin{aligned}
K_{1\gamma}^\delta K_{1\delta}^\gamma &= K_{11}^\delta K_{1\delta}^1 + K_{12}^\delta K_{1\delta}^2 \\
&= K_{11}^1 K_{11}^1 + K_{11}^2 K_{12}^1 + K_{12}^1 K_{11}^2 + K_{12}^2 K_{12}^2
\end{aligned}$$

$$\begin{aligned}
K_{2\gamma}^\delta K_{2\delta}^\gamma &= K_{21}^\delta K_{2\delta}^1 + K_{22}^\delta K_{2\delta}^2 \\
&= K_{21}^1 K_{21}^1 + K_{21}^2 K_{22}^1 + K_{22}^1 K_{21}^2 + K_{22}^2 K_{22}^2
\end{aligned}$$

$$\begin{aligned}
K_{1\gamma}^\delta K_{2\delta}^\gamma &= K_{11}^\delta K_{2\delta}^1 + K_{12}^\delta K_{2\delta}^2 \\
&= K_{11}^1 K_{21}^1 + K_{11}^2 K_{22}^1 + K_{12}^1 K_{21}^2 + K_{12}^2 K_{22}^2 .
\end{aligned}$$

This allows to establish a restriction on the value of Q using

$$\begin{aligned}
K_{1\gamma}^\delta K_{1\delta}^\gamma &\in \{1, 2\} \\
K_{1\gamma}^\delta K_{2\delta}^\gamma &\in \{0, 1, 2\} \\
K_{2\gamma}^\delta K_{2\delta}^\gamma &\in \{1, 2\}
\end{aligned}$$

and given that

$$K_{12}^1 K_{21}^2 = 0 \quad (a)$$

$$K_{11}^1 K_{21}^1 \neq 0 \quad \vee \quad K_{12}^2 K_{22}^2 \neq 0 \quad (b)$$

we have

$$b = K_{1\gamma}^\delta K_{2\delta}^\gamma \in \{0, 1\} . \quad (87)$$

This means that the condition is not satisfied for $a \cdot c < b^2$, when $a \neq 0 \neq c$. However this condition is satisfied when one of the diagonal elements are zero. So, we have

$$a \cdot c < b^2$$

leads to

$$1 \cdot 0 < 1$$

$$0 \cdot 1 < 1$$

$$2 \cdot 0 < 1$$

$$0 \cdot 2 < 1 .$$

If we consider the $K_{1\gamma}^\delta K_{1\delta}^\gamma$ object,

$$\begin{aligned} K_{1\gamma}^\delta K_{1\delta}^\gamma &= K_{11}^\delta K_{1\delta}^1 + K_{12}^\delta K_{1\delta}^2 \\ &= (K_{11}^1 K_{11}^1 = 0) + (K_{11}^2 K_{12}^1 \neq 0) + (K_{12}^1 K_{11}^2 \neq 0) + (K_{12}^2 K_{12}^2 = 0) \end{aligned} \quad (a)$$

or

$$\begin{aligned} K_{1\gamma}^\delta K_{1\delta}^\gamma &= K_{11}^\delta K_{1\delta}^1 + K_{12}^\delta K_{1\delta}^2 \\ &= (K_{11}^1 K_{11}^1 \neq 0) + (K_{11}^2 K_{12}^1 = 0) + (K_{12}^1 K_{11}^2 = 0) + (K_{12}^2 K_{12}^2 \neq 0) \end{aligned} \quad (b)$$

This is not true because the law of composition semigroup is univocally defined for each pair of elements. That is, it is not possible to find a semigroup of order two that allows us

to exchange parts of space V_+ with parts of space V_- . So we can say that "it is impossible to obtain $Q \neq 0$ using a semigroup S of order 2".

Consider now $H \in \{0, 1, 2\}$. Using the equality

$$(K_{1\gamma}^\delta K_{1\delta}^\gamma) (K_{2\gamma}^\delta K_{2\delta}^\gamma) = (K_{1\gamma}^\delta K_{2\delta}^\gamma)^2$$

$$a \cdot c = b^2$$

for

$$1 \cdot 1 = 1$$

$$2 \cdot 2 = 2^2$$

This means that $K_{1\gamma}^\delta K_{2\delta}^\gamma < 2$, because there is no way that more than two terms of this element is not null. So that

$$b < 2, \text{ i.e., } b \in \{0, 1\}$$

$$ac = b^2$$

$$1 \cdot 1 = 1 \tag{88}$$

and

$$1 \cdot 0 = 0 \cdot 1 = 0$$

$$2 \cdot 0 = 0 \cdot 2 = 0$$

But the last are forbidden. So we have that, for $H = 1$, the $K_{i\gamma}^\delta K_{j\delta}^\gamma$ matrix takes the form

$$(K_{i\gamma}^\delta K_{j\delta}^\gamma) = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

$$_d (K_{i\gamma}^\delta K_{j\delta}^\gamma) = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}.$$

To calculate the semigroup corresponding to this matrix M_K , we use

$$\begin{aligned}
K_{1\gamma}^\delta K_{1\delta}^\gamma &= 1 \\
K_{1\gamma}^\delta K_{2\delta}^\gamma &= 1 = K_{2\gamma}^\delta K_{1\delta}^\gamma \\
K_{2\gamma}^\delta K_{2\delta}^\gamma &= 1
\end{aligned} \tag{89}$$

We begin with the first double sum on a possible semigroup of the type $S_{2,1,0}$

$$\begin{aligned}
K_{1\gamma}^\delta K_{1\delta}^\gamma &= K_{11}^\delta K_{1\delta}^1 + K_{12}^\delta K_{1\delta}^2 \\
&= K_{11}^1 K_{11}^1 + K_{11}^2 K_{12}^1 + K_{12}^1 K_{11}^2 + K_{12}^2 K_{12}^2 \\
&= 1.
\end{aligned}$$

Choosing $K_{11}^1 K_{11}^1 = 1$ we have $K_{12}^2 K_{12}^2 = K_{12}^1 K_{11}^2 = K_{11}^2 K_{12}^1 = 0$. This implies that the law of composition of $S_{2,1,0}$ must satisfy

$$\begin{aligned}
s_1 \diamond s_1 &= s_1 \\
s_1 \diamond s_2 &\neq s_2 \longrightarrow s_1 \diamond s_2 = s_1
\end{aligned}$$

For the other double sum we have

$$\begin{aligned}
K_{2\gamma}^\delta K_{2\delta}^\gamma &= K_{21}^\delta K_{2\delta}^1 + K_{22}^\delta K_{2\delta}^2 \\
&= K_{21}^1 K_{21}^1 + K_{21}^2 K_{22}^1 + K_{22}^1 K_{21}^2 + K_{22}^2 K_{22}^2
\end{aligned}$$

which leads to

$$s_2 \diamond s_2 = s_1$$

and for non diagonal double sum (con $K_{11}^1 K_{11}^1 = 1$)

$$\begin{aligned}
K_{1\gamma}^\delta K_{2\delta}^\gamma &= K_{11}^\delta K_{2\delta}^1 + K_{12}^\delta K_{2\delta}^2 \\
&= K_{11}^1 K_{21}^1 + K_{11}^2 K_{22}^1 + K_{12}^1 K_{21}^2 + K_{12}^2 K_{22}^2 \\
&= K_{11}^1 K_{21}^1 + 0 + 0 + 0 \\
&= 1
\end{aligned}$$

This leads to the semigroup

$$\begin{array}{c} \diamond \lambda_1 \lambda_2 \\ S_{2,1,0} = \lambda_1 \lambda_1 \lambda_1 \\ \lambda_2 \lambda_1 \lambda_1 \end{array}$$

The $K_{12}^2 K_{12}^2 = 1$ condition

$$\begin{aligned} K_{1\gamma}^\delta K_{2\delta}^\gamma &= K_{11}^\delta K_{2\delta}^1 + K_{22}^\delta K_{2\delta}^2 \\ &= K_{11}^1 K_{21}^1 + K_{11}^2 K_{22}^1 + K_{12}^1 K_{21}^2 + K_{12}^2 K_{22}^2 \\ &= 0 + 0 + 0 + 1 \\ &= 1 \end{aligned}$$

leads to the semigroup

$$\begin{array}{c} \diamond \lambda_1 \lambda_2 \\ S_{2,1,0} = \lambda_1 \lambda_2 \lambda_2 \\ \lambda_2 \lambda_2 \lambda_2 \end{array}$$

Finally to $H = 0$ and $Q = 0$ one has the condition

$$\begin{aligned} (K_{1\gamma}^\delta K_{1\delta}^\gamma) (K_{2\gamma}^\delta K_{2\delta}^\gamma) &> (K_{1\gamma}^\delta K_{2\delta}^\gamma)^2 \\ ac &> b^2 \end{aligned}$$

we use again $K_{i\gamma}^\delta K_{j\delta}^\gamma \in \{0, 1, 2\}$ to find semigroups satisfying this condition.

$$\begin{aligned}
ac &> b^2 \\
2 \cdot 2 &> 1 \\
2 \cdot 1 &> 1 \\
1 \cdot 2 &> 1 \\
1 \cdot 1 &> 0 \\
2 \cdot 1 &> 0 \\
1 \cdot 2 &> 0 \\
2 \cdot 2 &> 0
\end{aligned} \tag{90}$$

This leads to the following matrices M_K

$$\begin{aligned}
(K_{i\gamma}^\delta K_{j\delta}^\gamma) &= \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \equiv M_{K1} \\
{}_d(K_{i\gamma}^\delta K_{j\delta}^\gamma) &= \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \\
&\wedge \\
(K_{i\gamma}^\delta K_{j\delta}^\gamma) &= \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \equiv M_{K2} \\
{}_d(K_{i\gamma}^\delta K_{j\delta}^\gamma) &= \begin{pmatrix} \frac{\sqrt{5}}{2} + \frac{3}{2} & 0 \\ 0 & \frac{3}{2} - \frac{\sqrt{5}}{2} \end{pmatrix} \\
&\wedge \\
(K_{i\gamma}^\delta K_{j\delta}^\gamma) &= \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \equiv M_{K3} \\
{}_d(K_{i\gamma}^\delta K_{j\delta}^\gamma) &= \begin{pmatrix} \frac{3}{2} - \frac{\sqrt{5}}{2} & 0 \\ 0 & \frac{\sqrt{5}}{2} + \frac{3}{2} \end{pmatrix} \\
&\wedge \\
(K_{i\gamma}^\delta K_{j\delta}^\gamma) &= \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \equiv M_{K4} \\
&\wedge \\
(K_{i\gamma}^\delta K_{j\delta}^\gamma) &= \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \equiv M_{K5} \\
&\wedge \\
(K_{i\gamma}^\delta K_{j\delta}^\gamma) &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \equiv M_{K6} \\
&\wedge \\
(K_{i\gamma}^\delta K_{j\delta}^\gamma) &= \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \equiv M_{K7}
\end{aligned}$$

To check whether there is any semigroup of order two that leads us to these matrices M_K

we follow a similar process to the one above case, with $K_{i\gamma}^\delta K_{j\delta}^\gamma$. If $K_{1\gamma}^\delta K_{1\delta}^\gamma = a = 2$ we have four possibilities

$$\begin{aligned}
ac &> b^2 \\
2 \cdot 2 &> 1 \\
2 \cdot 1 &> 1 \\
1 \cdot 2 &> 1 \\
2 \cdot 1 &> 0 \\
1 \cdot 2 &> 0 \\
2 \cdot 2 &> 0
\end{aligned}$$

corresponding to $K_{11}^1 K_{11}^1 \neq 0 \wedge K_{12}^2 K_{12}^2 \neq 0$ or $K_{11}^2 K_{12}^1 \neq 0$

$$\begin{aligned}
K_{1\gamma}^\delta K_{1\delta}^\gamma &= K_{11}^\delta K_{1\delta}^1 + K_{12}^\delta K_{1\delta}^2 \\
&= K_{11}^1 K_{11}^1 + K_{11}^2 K_{12}^1 \\
&\quad + K_{12}^1 K_{11}^2 + K_{12}^2 K_{12}^2 \\
&= 2
\end{aligned}$$

Choosing

$$\begin{aligned}
K_{11}^2 K_{12}^1 &\neq 0 \\
K_{11}^2 &\neq 0 \quad \wedge \quad K_{12}^1 &\neq 0
\end{aligned}$$

we find

$$\begin{aligned}
K_{2\gamma}^\delta K_{2\delta}^\gamma &= K_{21}^\delta K_{2\delta}^1 + K_{22}^\delta K_{2\delta}^2 \\
&= K_{21}^1 K_{21}^1 + K_{21}^2 K_{22}^1 \\
&\quad + K_{22}^1 K_{21}^2 + K_{22}^2 K_{22}^2 = \{1, 2\}
\end{aligned}$$

$$K_{22}^2 K_{22}^2 \in \{0, 1\}.$$

To $K_{22}^2 \neq 0$, $K_{2\gamma}^\delta K_{2\delta}^\gamma = 2$ we have

$$\begin{aligned}
K_{1\gamma}^\delta K_{2\delta}^\gamma &= K_{11}^\delta K_{2\delta}^1 + K_{12}^\delta K_{2\delta}^2 \\
&= K_{11}^1 K_{21}^1 + K_{11}^2 K_{22}^1 \\
&\quad + K_{12}^1 K_{21}^2 + K_{12}^2 K_{22}^2 \\
&= 0
\end{aligned}$$

$$K_{11}^1 K_{21}^1 = 0$$

$$K_{12}^1 K_{21}^2 = 0$$

$$K_{11}^2 K_{22}^1 = 0$$

$$K_{12}^2 K_{22}^2 = 0$$

So that

$$S_{2,0,0} = \begin{array}{|c|c|c|} \hline \diamond & \lambda_1 & \lambda_2 \\ \hline \lambda_1 & \lambda_2 & \lambda_1 \\ \hline \lambda_2 & \lambda_1 & \lambda_2 \\ \hline \end{array} \longrightarrow M_K = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}. \quad (91)$$

For choice $K_{22}^2 = 0$ we have $K_{1\gamma}^\delta K_{2\delta}^\gamma$ is non-zero, because $K_{11}^2 K_{22}^1 = 1$. With this choice, the set $K \neq 0 \longrightarrow \{K_{12}^1, K_{11}^2, K_{21}^1, K_{22}^2\}$ is obtained. The matrix M_K and semigroup in this case are

$$S_{2,0,0} = \begin{array}{|c|c|c|} \hline \diamond & \lambda_1 & \lambda_2 \\ \hline \lambda_1 & \lambda_2 & \lambda_1 \\ \hline \lambda_2 & \lambda_1 & \lambda_1 \\ \hline \end{array} \longrightarrow M_K = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \quad (92)$$

By choosing now, since the beginning $K_{11}^1 K_{11}^1 \neq 0$ and $K_{12}^2 K_{12}^2 \neq 0$, we have

$$\begin{aligned}
K_{2\gamma}^\delta K_{2\delta}^\gamma &= K_{21}^\delta K_{2\delta}^1 + K_{22}^\delta K_{2\delta}^2 \\
&= K_{21}^1 K_{21}^1 + K_{21}^2 K_{22}^1 + K_{22}^1 K_{21}^2 + K_{22}^2 K_{22}^2 \\
&= \{1, 2\}
\end{aligned}$$

is equal to two for $K_{22}^1 \neq 0$ and is equal to one for $K_{22}^1 = 0$ and $K_{22}^2 \neq 0$. Using $K_{22}^1 \neq 0$, $K_{2\gamma}^\delta K_{2\delta}^\gamma = 1$ we find

$$\begin{aligned} K_{1\gamma}^\delta K_{2\delta}^\gamma &= K_{11}^\delta K_{2\delta}^1 + K_{12}^\delta K_{2\delta}^2 \\ &= K_{11}^1 K_{21}^1 + K_{11}^2 K_{22}^1 + K_{12}^1 K_{21}^2 + K_{12}^2 K_{22}^2 \\ &= 0 \end{aligned}$$

So that:

$$S_{2,0,0} = \begin{array}{|c|c|c|} \hline \diamond & \lambda_1 & \lambda_2 \\ \hline \lambda_1 & \lambda_1 & \lambda_2 \\ \hline \lambda_2 & \lambda_2 & \lambda_1 \\ \hline \end{array} \longrightarrow M_K = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \quad (93)$$

Following the same procedure, we find the remaining possible semigroups. In summary, we have that the possible semigroups are

$$\begin{aligned} A &= \begin{array}{|c|c|c|} \hline \diamond & \lambda_1 & \lambda_2 \\ \hline \lambda_1 & \lambda_2 & \lambda_1 \\ \hline \lambda_2 & \lambda_1 & \lambda_2 \\ \hline \end{array} \longrightarrow M_{K_A} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} = \tilde{M}_{K_A} \longleftarrow \begin{array}{|c|c|c|} \hline \diamond & \lambda_1 & \lambda_2 \\ \hline \lambda_1 & \lambda_1 & \lambda_2 \\ \hline \lambda_2 & \lambda_2 & \lambda_1 \\ \hline \end{array} \\ B &= \begin{array}{|c|c|c|} \hline \diamond & \lambda_1 & \lambda_2 \\ \hline \lambda_1 & \lambda_2 & \lambda_1 \\ \hline \lambda_2 & \lambda_1 & \lambda_1 \\ \hline \end{array} \longrightarrow M_{K_B} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} = \tilde{M}_{K_B} \longleftarrow \begin{array}{|c|c|c|} \hline \diamond & \lambda_1 & \lambda_2 \\ \hline \lambda_1 & \lambda_1 & \lambda_2 \\ \hline \lambda_2 & \lambda_2 & \lambda_2 \\ \hline \end{array} \\ C &= \begin{array}{|c|c|c|} \hline \diamond & \lambda_1 & \lambda_2 \\ \hline \lambda_1 & \lambda_1 & \lambda_1 \\ \hline \lambda_2 & \lambda_1 & \lambda_2 \\ \hline \end{array} \longrightarrow M_{K_C} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \\ D &= \begin{array}{|c|c|c|} \hline \diamond & \lambda_1 & \lambda_2 \\ \hline \lambda_1 & \lambda_2 & \lambda_2 \\ \hline \lambda_2 & \lambda_2 & \lambda_1 \\ \hline \end{array} \longrightarrow M_{K_D} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \end{aligned}$$

$$(g_{AB})_{S_{2,0,0} \otimes so(3)} = -1 \cdot \begin{pmatrix} 2 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix} \quad (94)$$

Where, for example, A coincides with the cyclic group \mathbb{Z}_2

$$S_{2,0,0} \equiv A = \begin{array}{|c|c|c|} \hline \diamond & \lambda_1 & \lambda_2 \\ \hline \lambda_1 & \lambda_1 & \lambda_2 \\ \hline \lambda_2 & \lambda_2 & \lambda_1 \\ \hline \end{array} \longrightarrow \begin{array}{|c|c|c|} \hline \diamond & \lambda_0 & \lambda_1 \\ \hline \lambda_0 & \lambda_0 & \lambda_1 \\ \hline \lambda_0 & \lambda_1 & \lambda_0 \\ \hline \end{array} = \mathbb{Z}_2 . \quad (95)$$

The natural question is: multiplication tables B , C , D are semigroups multiplication tables?. To answer let us see if these multiplication satisfy associativity. It is direct to verify that multiplication tables B and D do not satisfy the associative property. So that the associated semigroup table C lead to a Lie algebra, and semisimple compact, like \mathbb{Z}_2 semigroup. The question now is whether the algebras obtained by expansion using \mathbb{Z}_2 and C are isomorphic or not. The answer can be found in two ways. The first is to use the character table shown above. From this table we can verify by inspection that $\chi = -6$ univocally characterizes $so(4)$ and its isomorphic forms.

The other way to tell if the two Lie algebras are isomorphic or not, is using *MAGMA* [21]. With this program you can check if the product spaces, are or are not Lie algebras and thus check whether the tables obtained correspond to semigroups.

The above outcomes allow to state: "The semigroup which leads from a Lie algebra to another by S-expansion method is not necessarily unique."

Following the procedure described above is possible to obtain the possible S-expansions that could lead from $so(n)$ to $so(n+l)$.

$so(n)$	$so(n+l)$	P	H	Q
3	4	2	0	0
3	6	5	0	0
3	7	7	0	0
3	9	12	0	0
3	10	15	0	0
3	12	22	0	0
4	9	6	0	0
4	12	11	0	0
4	13	13	0	0
4	16	20	0	0
5	16	12	0	0
5	20	19	0	0
5	21	21	0	0
6	10	3	0	0
6	15	7	0	0
6	16	8	0	0

(*Tabla I*)

V. NO-SIMPLICITY OF S-EXPANDED ALGEBRA

From the above table we can see that there are several possible S-expansions that we could perform. However there are some restrictions that must be satisfied. A simple Lie algebra is one that has no nontrivial ideals and that can not be expressed as a direct sum of other Lie algebras. This fact leads to the important result that an Lie algebra obtained by S -expansion of another Lie algebra can not be simple.

A square matrix $(M_f)_S$ of order P and range P , will have P nonzero eigenvalues and can be expressed in the form

$$(M_f)_{S,d} = \begin{pmatrix} \lambda_0 & & O \\ & \ddots & \\ O & & \lambda_{P-1} \end{pmatrix}_{P \times P} . \quad (96)$$

The Kronecker product between $(M_f)_{S,d}$ and an arbitrary order matrix R such as $Ad(\mathcal{G})$ is given by

$$\begin{aligned} \begin{pmatrix} \lambda_0 & & O \\ & \ddots & \\ O & & \lambda_{P-1} \end{pmatrix}_{P \times P} \otimes Ad(\mathcal{G}) &= \begin{pmatrix} \lambda_0(\bar{M}) & & O \\ & \ddots & \\ O & & \lambda_{P-1}(\bar{M}) \end{pmatrix}_{PR \times PR} \\ &= \underbrace{\lambda_0(\bar{M}) \oplus \cdots \oplus \lambda_{P-1}(\bar{M})}_{P\text{-times}} \equiv Ad(\bar{\mathcal{G}}) \end{aligned} \quad (97)$$

This means that if (\bar{M}) is the adjoint representation of an arbitrary Lie algebra \mathcal{G} and if $(M_f)_S$ is a faithful matrix representation of an abelian, discrete and finite semigroup S , then $Ad(\bar{\mathcal{G}})$ is the adjoint representation of a non-simple Lie algebra, given by the direct sum of P Lie algebras \mathcal{G} (which can be simple or not). This will occur when the rank of the matrix $(M_f)_S$ is equal to the number of elements of the semigroup S .

This result leads to state the following

Theorem: If S is a finite, discrete and abelian semigroup and if \mathcal{G} is an arbitrary Lie algebra, then the product space $S \otimes \mathcal{G}$ is a non-simple Lie algebra consisting of the direct sum of P original Lie algebras \mathcal{G} , where P is the number of elements of the semigroup S .

Proof: The faithful matrix representation of any abelian, discrete and finite semigroup has the form

$$(M_f)_S = \begin{pmatrix} (K_r)_0^0 & \cdots & (K_u)_0^x & \cdots & (K_x)_0^{P-1} \\ \vdots & \ddots & & & \vdots \\ (K_s)_i^0 & \cdots & (K_v)_i^x & & (K_y)_i^{P-1} \\ \vdots & & & \ddots & \vdots \\ (K_t)_{p-1}^0 & \cdots & (K_w)_i^x & \cdots & (K_z)_{P-1}^{P-1} \end{pmatrix}_{P \times P}. \quad (98)$$

If the matrix has a lower rank than P then will have (i) two equal or proportional rows or alternatively (ii) a third row that is a linear combination of other linearly independent rows.

(i) **Two equal or proportional rows:** We will use the indices i, j, k, r with $j \neq r$ (e.g. $j < r$). The operation between the i -th and j -th element of the semigroup results in the k -th element of the semigroup. The corresponding matrix element is $(K_i)_j^k$, which

is located in the j -th row and k -th column of the matrix $(M_f)_S$. If there is an i -th row equal or proportional to it, then there is also an element of the form $C(K_j)_i^r$, which is not in the k -th column because if the rows were equal, then the element in the row belonging to the k -th column will have the form $(K_i)_i^k$, for which $k \neq r$. This has the consequence that $\lambda_i \diamond \lambda_j = \lambda_k$ and $\lambda_j \diamond \lambda_i = \lambda_r$ with $\lambda_k \neq \lambda_r$ implying that $\lambda_i \diamond \lambda_j \neq \lambda_j \diamond \lambda_i$.

This contradicts the condition of abelian semigroup S . So the matrix $(M_f)_S$ has no equal (or proportional) rows. Similarly it is proved that the matrix $(M_f)_S$ does not have equal (or proportional) columns.

(b) A row is a linear combination of linearly independent rows: Note that the j -th element of the i -th row has the form $(K_j)_i^y$. This means that if another element in the same row has the form $(K_j)_i^z$ to different columns, i.e., for $y \neq z$, then this implies $\lambda_j \diamond \lambda_i = \lambda_y$ y $\lambda_j \diamond \lambda_i = \lambda_z$. But $\lambda_y \neq \lambda_z$. This has the consequence that $\lambda_j \diamond \lambda_i \neq \lambda_j \diamond \lambda_i$, which is absurd. Additionally this would imply that the internal binary operation \diamond semigroup is not univocally defined for each pair of elements. This allows us to ensure that we never have two or more $(K_j)_i^y$ -selectores associated with the same binary operation $\lambda_j \diamond \lambda_i$ in a row.

Thus we have the row that is generated by the linear combination of at least two independent rows in the matrix $(M_f)_S$ will always have an element of the form

$$C^{i_0} (K_q)_{i_0}^y + \dots + C^{i_{P-1}} (K_x)_{i_{P-1}}^y \quad (99)$$

and another element of the form

$$C^{i_0} (K_q)_{i_0}^z + \dots + C^{i_{P-1}} (K_x)_{i_{P-1}}^z. \quad (100)$$

So that for the elements of the semigroup one has

$$\begin{aligned} \lambda_q \diamond \lambda_{i_0} = \lambda_y \quad \wedge \quad \lambda_q \diamond \lambda_{i_0} = \lambda_z \\ \vdots \\ \lambda_x \diamond \lambda_{i_{P-1}} = \lambda_y \quad \wedge \quad \lambda_x \diamond \lambda_{i_{P-1}} = \lambda_z, \end{aligned} \quad (101)$$

i.e.,

$$\begin{aligned}
\lambda_q \diamond \lambda_{i_0} &\neq \lambda_q \diamond \lambda_{i_0} \\
&\vdots \\
\lambda_x \diamond \lambda_{i_{P-1}} &\neq \lambda_x \diamond \lambda_{i_{P-1}}.
\end{aligned} \tag{102}$$

This leads to the absurd result that there are more than one internal binary operations that are not univocally defined.

These results allow us to affirm that the faithful matrix representation $(M_f)_S$ of a semi-group of P elements will always be of rank P . Therefore, the diagonal form will have P nonzero eigenvalues and the product space $S \otimes \mathcal{G}$ will be the direct sum of P times the original Lie algebra \mathcal{G} . This means that $S \otimes \mathcal{G}$ will be a non-simple Lie algebra.

VI. CONCLUDING REMARKS

In this work we have reviewed some concepts of the theory of Lie algebras and the main aspects of the S-expansion procedure. Probably the most important result of this article is the fact that the S-expansion procedure affects the geometry of a Lie group: was found how changing the magnitude of a vector and the angle between two vectors. Was outlined, via an example, a method for determining the semigroup, which would provide a Lie algebra from another and then proved that a Lie algebra obtained from another Lie algebra via S-expansion is a non-simple Lie algebra.

A future work could be consider the geométrical analysis of the 0_S -Resonant procedure and also get the selection rules to determine when it's possible to get another algebra and then obtain the suitable semigroups and the partitions that are necessary to obtain it finally in each case (working progress).

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VII. APPENDIX A: INNER PRODUCT IN THE S-EXPANDED LIE ALGEBRA

Let $\{\lambda_\alpha\}$ be an abelian semigroup with two-selectors $K_{\alpha\beta}^\gamma$ and \mathfrak{g} a Lie algebra with basis $\{T_A\}$ and structure constants C_{AB}^C . Denote a basis elements of the S -expanded Lie algebra $S \otimes \mathfrak{g}$ by $T_{(a,\alpha)} \equiv \lambda_\alpha T_a$. The inner product between the $X = v^{(\alpha,a)} T_{(a,\alpha)}$ vectors of the $S \otimes \mathfrak{g}$ space is given by

$$(X, X)_{S\text{-exp}} \equiv \text{tr} (R(X) R(X)) = v^{(\alpha,a)} v^{(\beta,b)} K_{\alpha\gamma}^\delta K_{\beta\delta}^\gamma (X_a, X_b) \quad (103)$$

where $v^{(\alpha,a)}$ are the $S \otimes \mathfrak{g}$ coordinates, $K_{\beta\delta}^\gamma$, the K -selectors defined by product between the semigroups elements and (X_a, X_b) is the Killing-Cartan inner product defined in the Lie algebra \mathfrak{g} . Now we will show that the product (103) satisfy the axioms of the inner product.

In fact,

$$(a) \quad \text{If } X, Y, Z \in S \otimes \mathfrak{g}, \text{ then } (X + Y, Z) = (X, Z) + (Y, Z)$$

Proof: Since,

$$\begin{aligned} (X + Y, Z)_{S\text{-exp}} &= \text{tr} (R(X + Y) R(Z)) \\ &= \text{tr} (R(X) R(Z) + R(Y) R(Z)) \end{aligned}$$

we have

$$\begin{aligned} (X + Y, Z)_{S\text{-exp}} &= \text{tr} (v^{(\alpha,a)} R(X_{(\alpha,a)}) v^{(\gamma,c)} R(Z_{(\gamma,c)})) \\ &\quad + \text{tr} (v^{(\beta,b)} R(Y_{(\beta,b)}) v^{(\gamma,c)} R(Z_{(\gamma,c)})) \\ &= v^{(\alpha,a)} v^{(\gamma,c)} K_{\alpha\delta}^\varepsilon K_{\gamma\varepsilon}^\delta (C_a)_d^e (C_c)_e^d \\ &\quad + v^{(\beta,b)} v^{(\gamma,c)} K_{\beta\delta}^\varepsilon K_{\gamma\varepsilon}^\delta (C_b)_d^e (C_c)_e^d \\ &= v^{(\alpha,a)} v^{(\gamma,c)} K_{\alpha\delta}^\varepsilon K_{\gamma\varepsilon}^\delta \text{tr} (R(X_a) R(Z_c)) \\ &\quad + v^{(\beta,b)} v^{(\gamma,c)} K_{\beta\delta}^\varepsilon K_{\gamma\varepsilon}^\delta \text{tr} (R(Y_b) R(Z_c)) \\ &= v^{(\alpha,a)} v^{(\gamma,c)} K_{\alpha\delta}^\varepsilon K_{\gamma\varepsilon}^\delta (X_a, Z_c) \\ &\quad + v^{(\beta,b)} v^{(\gamma,c)} K_{\beta\delta}^\varepsilon K_{\gamma\varepsilon}^\delta (Y_b, Z_c) \\ &= (X, Z)_{S\text{-exp}} + (Y, Z)_{S\text{-exp}} \quad , \end{aligned}$$

where we have used (i) linearity of the adjoint representation, (ii) linearity of the trace, (iii) the definition of the Killing-Cartan inner product in the algebra \mathfrak{g} .

(b) If $X, Y \in S \otimes \mathfrak{g}$, then $(\alpha X, Y) = \alpha (X, Y)$

Proof:

$$\begin{aligned}
(\alpha X, Y)_{S-\text{exp}} &= \text{tr} \left(v^{(\alpha, a)} R \left(\alpha X_{(\alpha, a)} \right) v^{(\beta, b)} R \left(Y_{(\beta, b)} \right) \right) \\
&= \alpha v^{(\alpha, a)} v^{(\beta, b)} K_{\alpha\gamma}^{\delta} K_{\beta\delta}^{\gamma} (C_a)_c^d (C_b)_d^c \\
&= \alpha v^{(\alpha, a)} v^{(\beta, b)} K_{\alpha\gamma}^{\delta} K_{\beta\delta}^{\gamma} (X_a, X_b) \\
&= \alpha (X, Y)_{S-\text{exp}} \quad ,
\end{aligned}$$

where we have used (i) property of the trace, (ii) the definition of the Killing-Cartan inner product in the algebra \mathfrak{g} .

(c) If $X, Y \in S \otimes \mathfrak{g}$, then $(X, Y) = (Y, X)$

Proof:

$$\begin{aligned}
(X, Y)_{S-\text{exp}} &= \text{tr} (R(X) R(Y)) = \text{tr} \left(v^{(\alpha, a)} R \left(X_{(\alpha, a)} \right) v^{(\beta, b)} R \left(Y_{(\beta, b)} \right) \right) \\
&= v^{(\alpha, a)} v^{(\beta, b)} K_{\alpha\gamma}^{\delta} (C_a)_c^d K_{\beta\delta}^{\gamma} (C_b)_d^c \\
&= v^{(\alpha, a)} v^{(\beta, b)} K_{\alpha\gamma}^{\delta} K_{\beta\delta}^{\gamma} \text{tr} (R(X_a) R(Y_b)) \\
&= v^{(\alpha, a)} v^{(\beta, b)} K_{\alpha\gamma}^{\delta} K_{\beta\delta}^{\gamma} \text{tr} (R(Y_b) R(X_a)) \\
&= v^{(\beta, b)} v^{(\alpha, a)} K_{\beta\delta}^{\gamma} K_{\alpha\gamma}^{\delta} (X_b, X_a) \\
&= (Y, X)_{S-\text{exp}} \quad ,
\end{aligned}$$

where we have used the fact that (i) $R(X_a)$ and $R(Y_b)$ are square matrices, (ii) $(K_{\beta})_{\delta}^{\gamma}$ are square matrices and forms a faithful representation of the semigroup elements, so that $K_{\alpha\gamma}^{\delta} K_{\beta\delta}^{\gamma} = K_{\beta\delta}^{\gamma} K_{\alpha\gamma}^{\delta}$.

VIII. APPENDIX B: INVARIANCE OF THE PRODUCT $(X, Y)_{S-\text{exp}}$

The scalar product of arbitrary two elements $X = X^a T_a$ and $Y = Y^b T_b$ of the finite-dimensional Lie algebra g is given by the Killing form

$$(X, Y) := \text{Tr}(adX, adY) \tag{104}$$

We know that scalar product is invariant under the action of the Lie group G , if

$$(gXg^{-1}, gYg^{-1}) = (X, Y) \quad (105)$$

where g is an element of the G group. For infinitesimal case it is equivalent to

$$([X, Y], Z) = (X, [Y, Z]). \quad (106)$$

We will use this condition to show that the Killing-Cartan product $(X, X)_{S-\exp}$ is invariant under the action of the Lie group $S \times G$, that is we will use $([X, Y], Z)_{S-\exp} = (X, [Y, Z])_{S-\exp}$. This means that we must show that

$$Tr(ad[X_{(\alpha,A)}, X_{(\beta,B)}] \cdot adX_{(\gamma,C)}) = Tr(adX_{(\alpha,A)} \cdot ad[X_{(\beta,B)}, X_{(\gamma,C)}]) \quad (107)$$

where $X_{(\alpha,A)} = \lambda_\alpha X_A$.

Since,

$$[X_{(\alpha,A)}, X_{(\beta,B)}] = C_{(\alpha,A)(\beta,B)}^{(\gamma,C)} X_{(\gamma,C)}, \quad (108)$$

we have

$$ad[X_{(\alpha,A)}, X_{(\beta,B)}] = C_{(\alpha,A)(\beta,B)}^{(\gamma,C)} adX_{(\gamma,C)}, \quad (109)$$

so that

$$C_{(\alpha,A)(\beta,B)}^{(\delta,D)} Tr [(ad(X_{(\delta,D)}) \cdot ad(X_{(\gamma,C)}))] = C_{(\beta,B)(\gamma,C)}^{(\delta,D)} Tr [(adX_{(\alpha,A)} \cdot adX_{(\delta,D)})]. \quad (110)$$

Taking into account that

$$ad(X_{(\gamma,C)}) = C_{(\gamma,C)(\rho,L)}^{(\lambda,J)}, \quad (111)$$

we can see that

$$C_{(\alpha,A)(\beta,B)}^{(\delta,D)} C_{(\delta,D)(\epsilon,E)}^{(\epsilon,F)} C_{(\gamma,C)(\epsilon,F)}^{(\epsilon,E)} = C_{(\beta,B)(\gamma,C)}^{(\delta,D)} C_{(\alpha,A)(\epsilon,E)}^{(\epsilon,F)} C_{(\delta,D)(\epsilon,F)}^{(\epsilon,E)}. \quad (112)$$

Since $C_{(\alpha,A)(\beta,B)}^{(\delta,D)} = K_{\alpha\beta}^\delta C_{AB}^{..D}$, we can write

$$K_{\alpha\beta}^\delta K_{\delta\epsilon}^\epsilon K_{\gamma\epsilon}^\epsilon C_{AB}^{..D} C_{DE}^{..F} C_{CF}^{..E} = K_{\beta\gamma}^\delta K_{\alpha\epsilon}^\epsilon K_{\delta\epsilon}^\epsilon C_{BC}^{..D} C_{AE}^{..F} C_{DF}^{..E}.$$

But since the Killing-Cartan product in \mathfrak{g} is invariant under transformations in \mathfrak{g} , i.e.,

$$C_{AB}^{..D} C_{DE}^{..F} C_{CF}^{..E} = C_{BC}^{..D} C_{AE}^{..F} C_{DF}^{..E}. \quad (113)$$

This means that the invariance of Killing-Cartan product $(X, X)_{S-\text{exp}}$ leads to the following condition condition S for semigroup:

$$K_{\alpha\beta}^{\delta} K_{\delta\epsilon}^{\epsilon} K_{\gamma\epsilon}^{\epsilon} = K_{\beta\gamma}^{\delta} K_{\alpha\epsilon}^{\epsilon} K_{\delta\epsilon}^{\epsilon}. \quad (114)$$

On the other hand, in Ref. [5] was shown that the n -selectors 2-selectors $K_{\alpha\beta}^{\gamma}$ of the S semigroup satisfy the properties,

$$K_{\alpha_1 \dots \alpha_{n-1}}^{\sigma} K_{\sigma \alpha_n}^{\rho} = K_{\alpha_1 \sigma}^{\rho} K_{\alpha_2 \dots \alpha_n}^{\sigma} = K_{\alpha_1 \alpha_2 \dots \alpha_n}^{\rho}, \quad (115)$$

$$K_{\alpha\beta}^{\delta} K_{\delta\epsilon}^{\epsilon} = K_{\alpha\delta}^{\epsilon} K_{\beta\epsilon}^{\delta} = K_{\alpha\beta\epsilon}^{\epsilon}, \quad (116)$$

$$K_{\alpha\beta}^{\delta} K_{\delta\epsilon}^{\epsilon} = K_{\alpha\delta}^{\epsilon} K_{\beta\epsilon}^{\delta} = K_{\alpha\delta}^{\epsilon} K_{\epsilon\beta}^{\delta} = K_{\alpha\epsilon\beta}^{\epsilon},$$

$$K_{\beta\alpha\epsilon}^{\epsilon} = K_{\alpha\beta\epsilon}^{\epsilon} = K_{\alpha\epsilon\beta}^{\epsilon}.$$

Using the above properties we find

$$K_{\alpha\beta}^{\delta} K_{\delta\epsilon}^{\epsilon} K_{\gamma\epsilon}^{\epsilon} = K_{\alpha\beta\epsilon}^{\epsilon} K_{\gamma\epsilon}^{\epsilon} = K_{\alpha\beta\epsilon}^{\epsilon} K_{\epsilon\gamma}^{\epsilon} = K_{\alpha\beta\epsilon\gamma}^{\epsilon} = K_{\alpha\beta\gamma\epsilon}^{\epsilon} \quad (117)$$

$$K_{\beta\gamma}^{\delta} K_{\alpha\epsilon}^{\epsilon} K_{\delta\epsilon}^{\epsilon} = K_{\beta\gamma}^{\delta} K_{\alpha\delta\epsilon}^{\epsilon} = K_{\beta\gamma}^{\delta} K_{\delta\alpha\epsilon}^{\epsilon} = K_{\beta\gamma\alpha\epsilon}^{\epsilon} = K_{\beta\alpha\gamma\epsilon}^{\epsilon} = K_{\alpha\beta\gamma\epsilon}^{\epsilon} \quad (118)$$

and therefore

$$K_{\alpha\beta}^{\delta} K_{\delta\epsilon}^{\epsilon} K_{\gamma\epsilon}^{\epsilon} = K_{\beta\gamma}^{\delta} K_{\alpha\epsilon}^{\epsilon} K_{\delta\epsilon}^{\epsilon}. \quad (119)$$

Comparing the result (114) with (119), we conclude that the inner product of Killing-Cartan $(X, X)_{S-\text{exp}}$ is invariant under linear transformations generated by $S \otimes \mathcal{G}$.

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